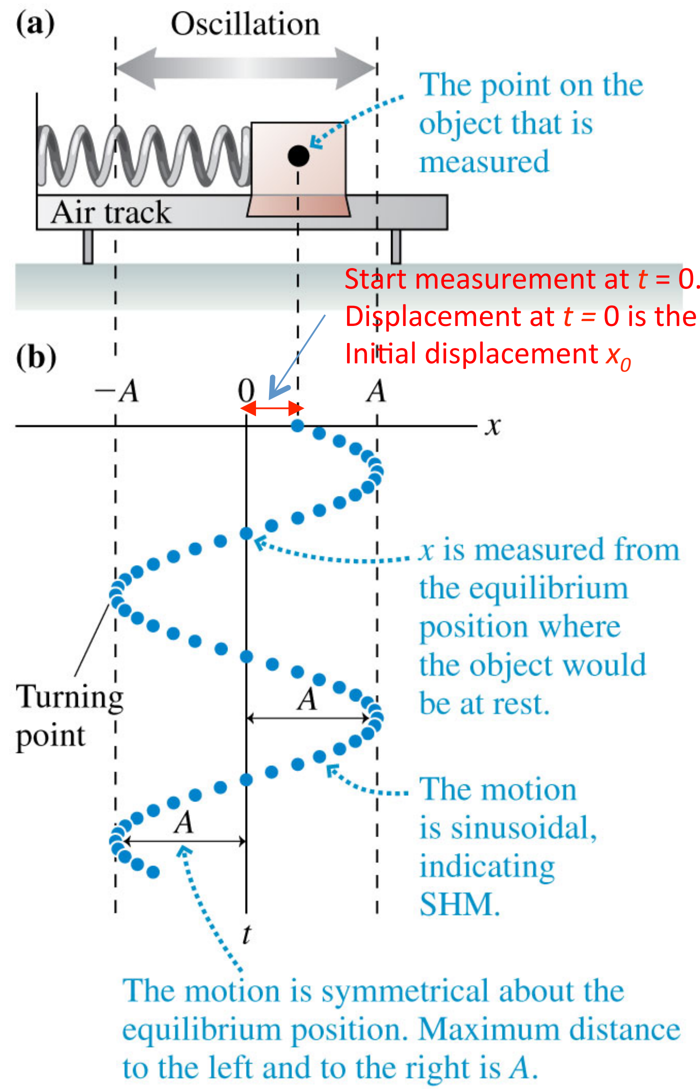
SHM – Initial displacement and Phase constant ϕ_0

- Remember, SHM is **simple harmonic motion**.
- In figure (a) an air-track glider is attached to a spring.
- Figure (b) shows the glider's position measured 20 times every second.
- SHM: $x(t) = A\cos(\omega t + \phi_0)$

  At t = 0, $x(t=0) = x_0 = A\cos\phi_0$

  So...Initial Phase $\phi_0 = \cos^{-1}(x_0 / A)$

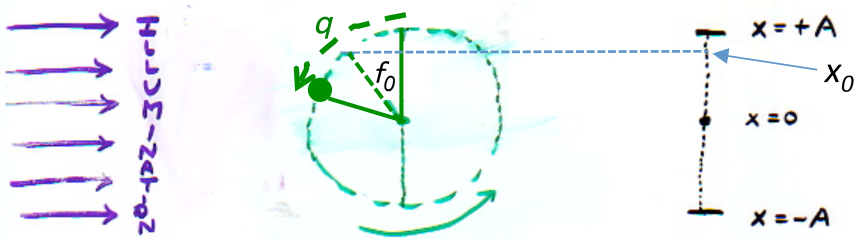


Important insight into ω ... continued. And ϕ_0 !

LINK BETWEEN ROTATIONAL MOTION & OSCILLATORY MOTION

or

LINK BETWEEN ANGULAR SPEED & ANGULAR FREQUENCY } DIFFERENT NAMES, SAME SYMBOL ω



mass rotating in a circle of radius A with angular speed ω

shadow oscillating up & down with angular frequency ω

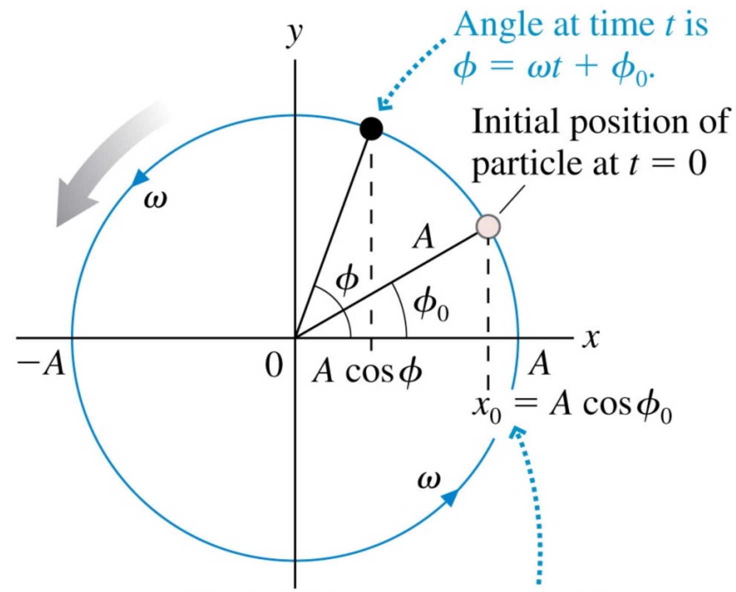
T = Time period for 1 revolution of rotating mass = Time period for 1 full oscillation of oscillating shadow

$f = \frac{1}{T}$ = Frequency = # of revs/sec of rotating mass = # of oscillations/sec of oscillating shadow

$\frac{2\pi}{T} = \omega = \omega$

$\omega = 2\pi f$ Relation between Angular Frequency ω & Frequency f

UNITS: T (secs) ; f (s^{-1} or Hz) ; ω (rad s^{-1})

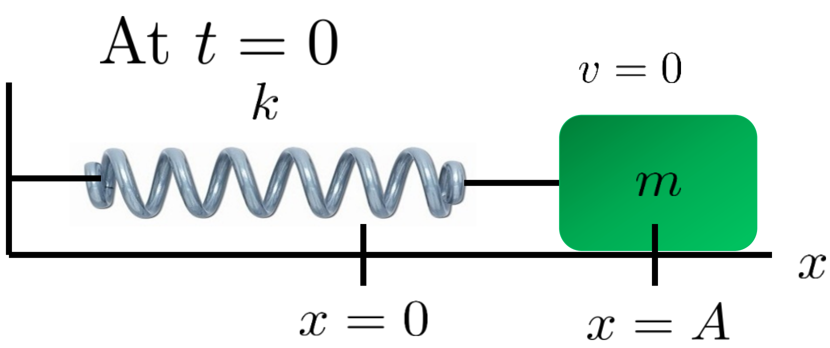


The initial x-component of the particle's position can be anywhere between $-A$ and A, depending on ϕ_0.

Let's watch a video…

Finding the Phase Constant from the Initial Conditions

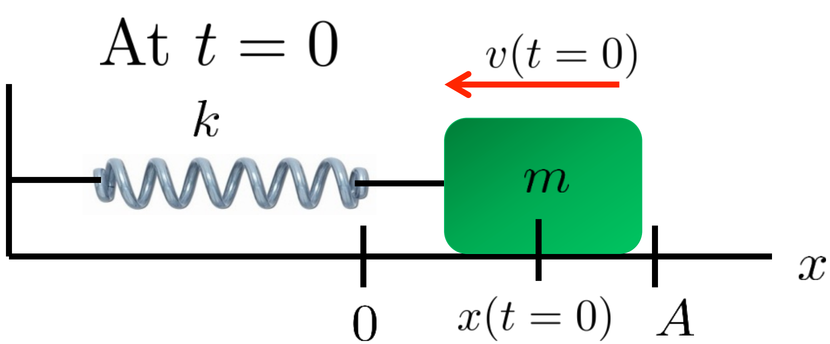
Recall what we had for the basic SHO:



If the block is pulled to $x = +A$ and released from rest at $t = 0$, then $A =$ the amplitude and $\phi_0 = 0$.

$$x(t) = A\cos(\omega t)$$

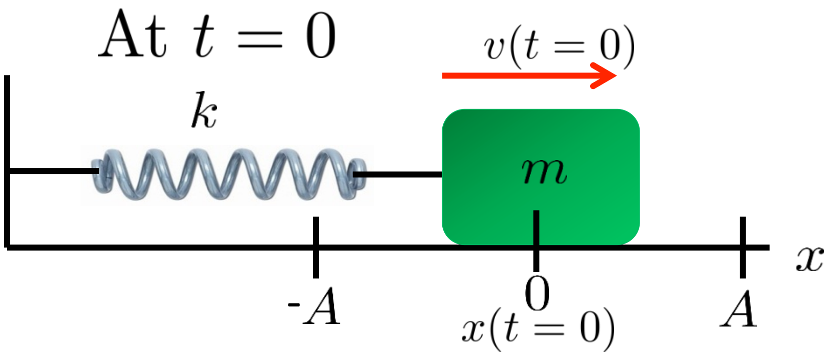
What if we have something like this for the initial conditions?



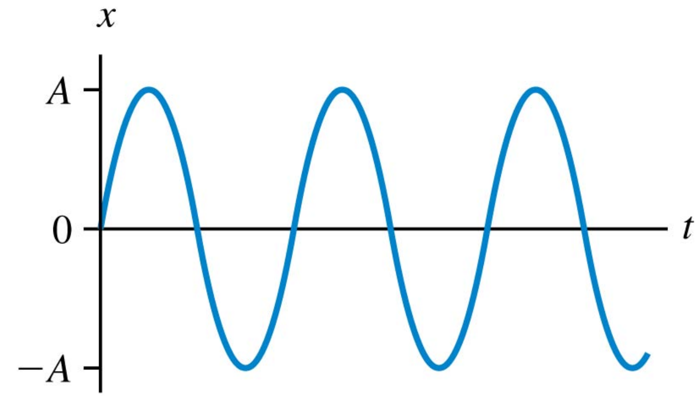
The block still oscillates between $\pm A$, but now $\phi \neq 0$, so:

$$x(t) = A\cos(\omega t + \phi_0)$$

How can we use the initial position and velocity to find the phase constant?



This is the position graph of a mass oscillating on a horizontal spring. What is the phase constant ϕ_0?



A. $-\pi/2$ rad.

B. 0 rad.

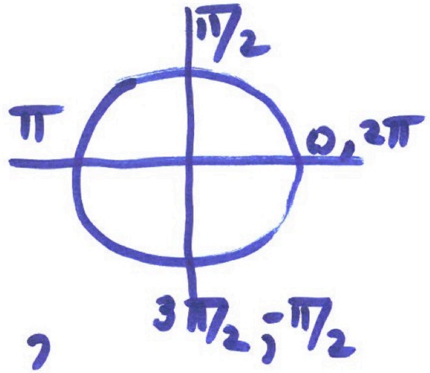
C. $\pi/2$ rad.

D. π rad.

E. None of these.

Q: What if the velocity vector above was pointing to the **left** at $t = 0$, instead of right? **What would ϕ_0 be then?**

$$X(t) = A\cos(\omega t + \phi_0)$$



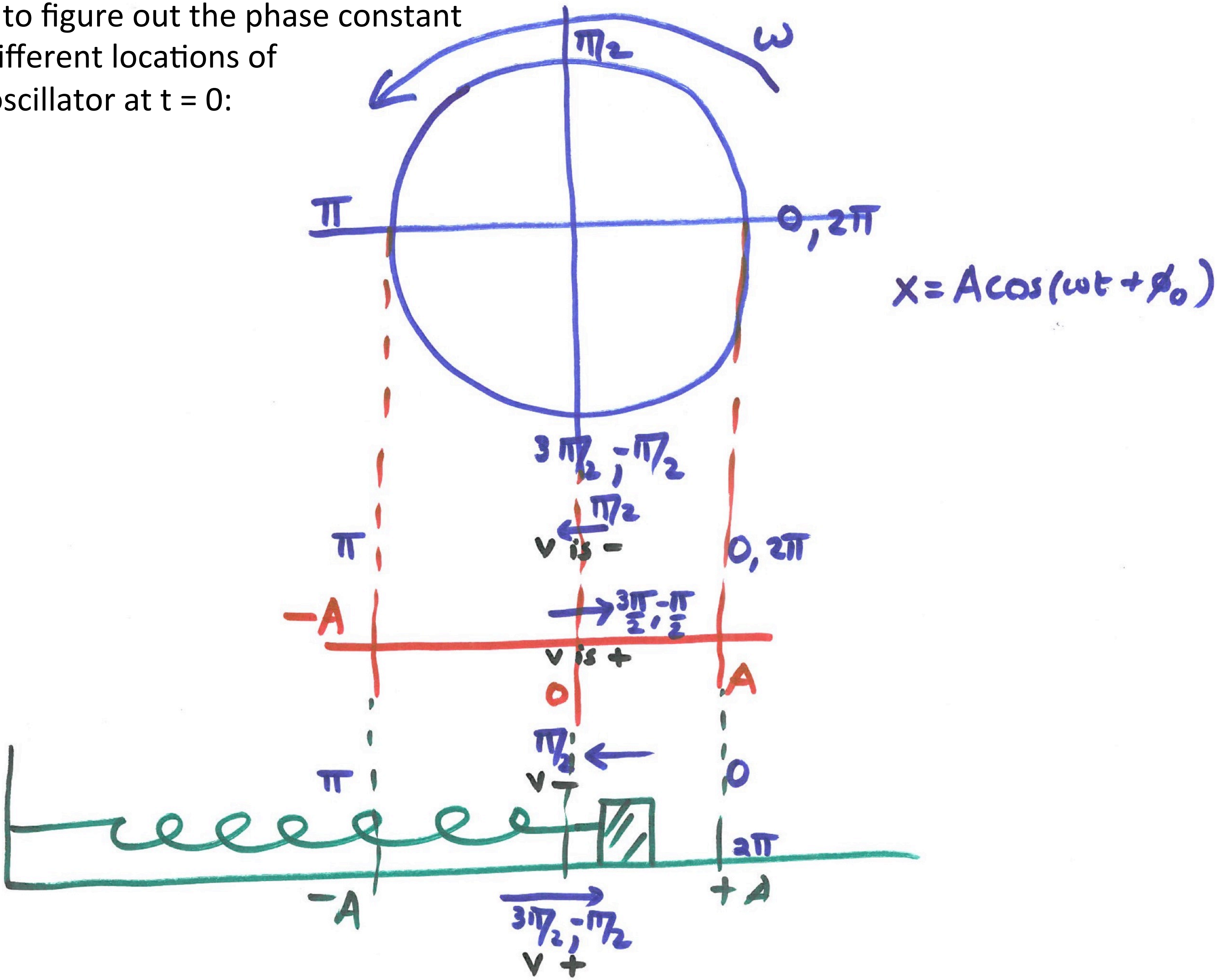
At $t=0$, $X=0$

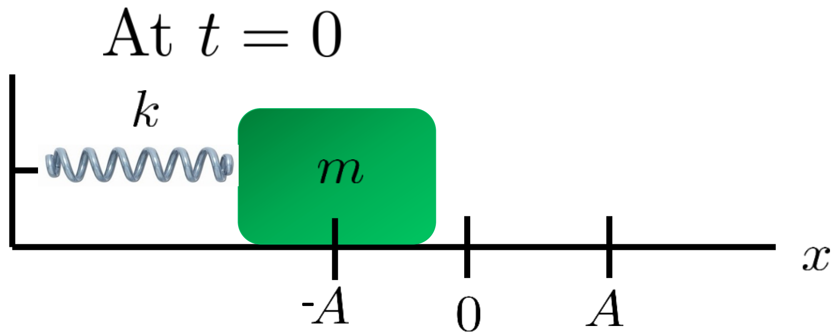
$$\therefore 0 = A\cos(\phi_0)$$

$$\Rightarrow \phi_0 = \pi/2 \quad \text{or} \quad 3\pi/2\,? \quad -\pi/2\,?$$

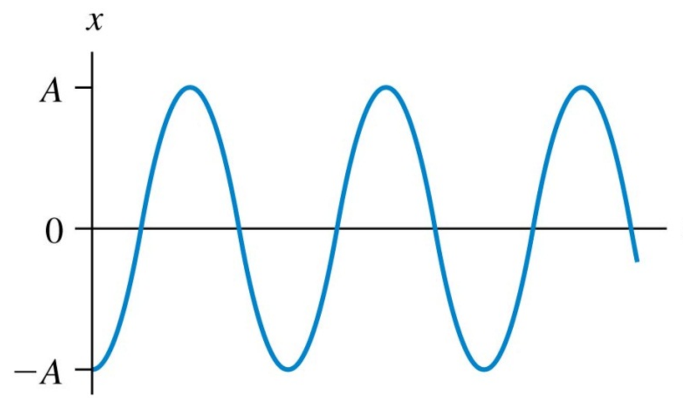
To determine which is the correct solution, I need to know what the velocity (direction, really) is @ $t=0$.

How to figure out the phase constant
for different locations of
the oscillator at t = 0:





This is the position graph of a mass oscillating on a horizontal spring. What is the phase constant ϕ_0?



A. **-π** /2 rad.

B. 0 rad.

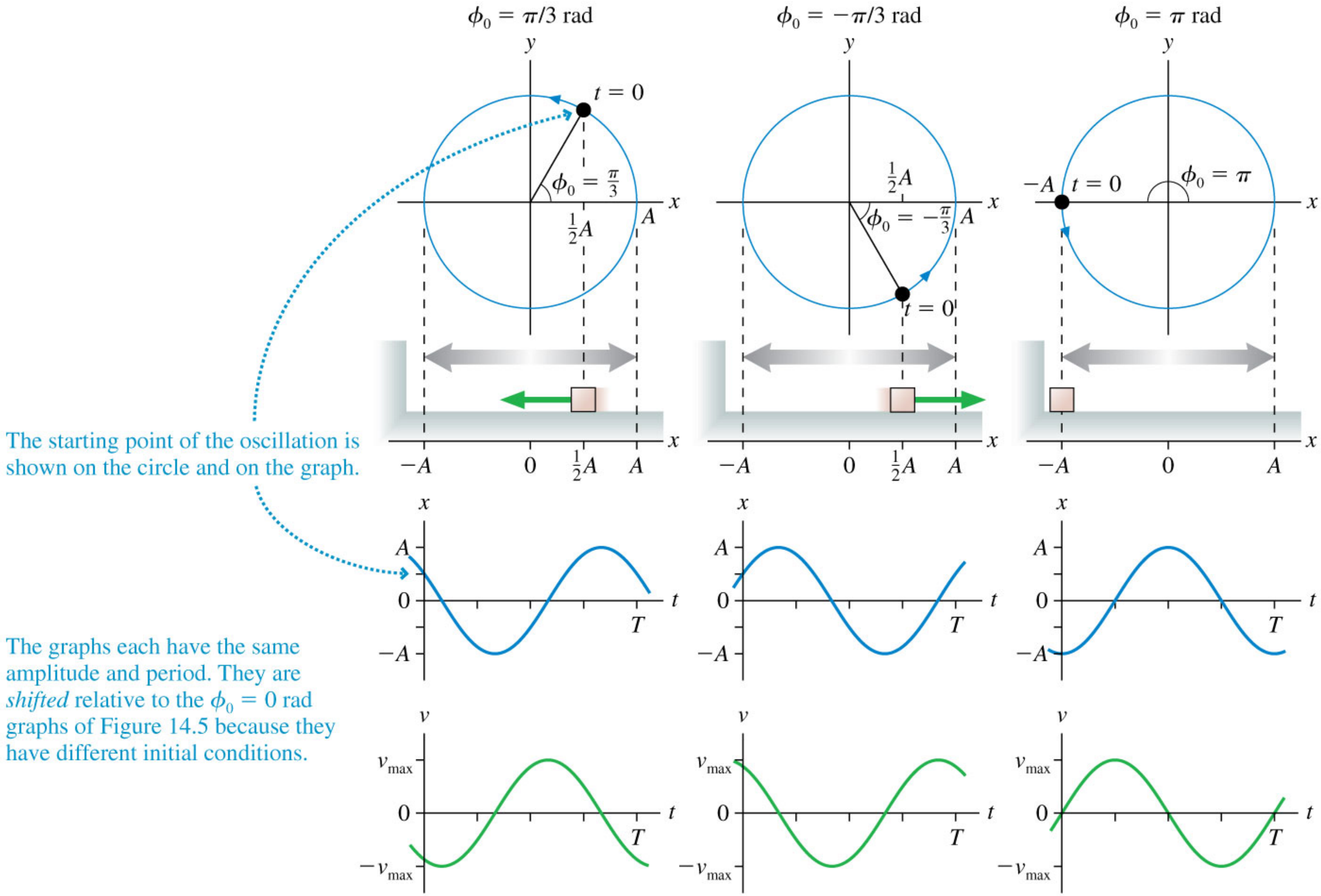
C. π /2 rad.

D. π rad.

E. None of these.

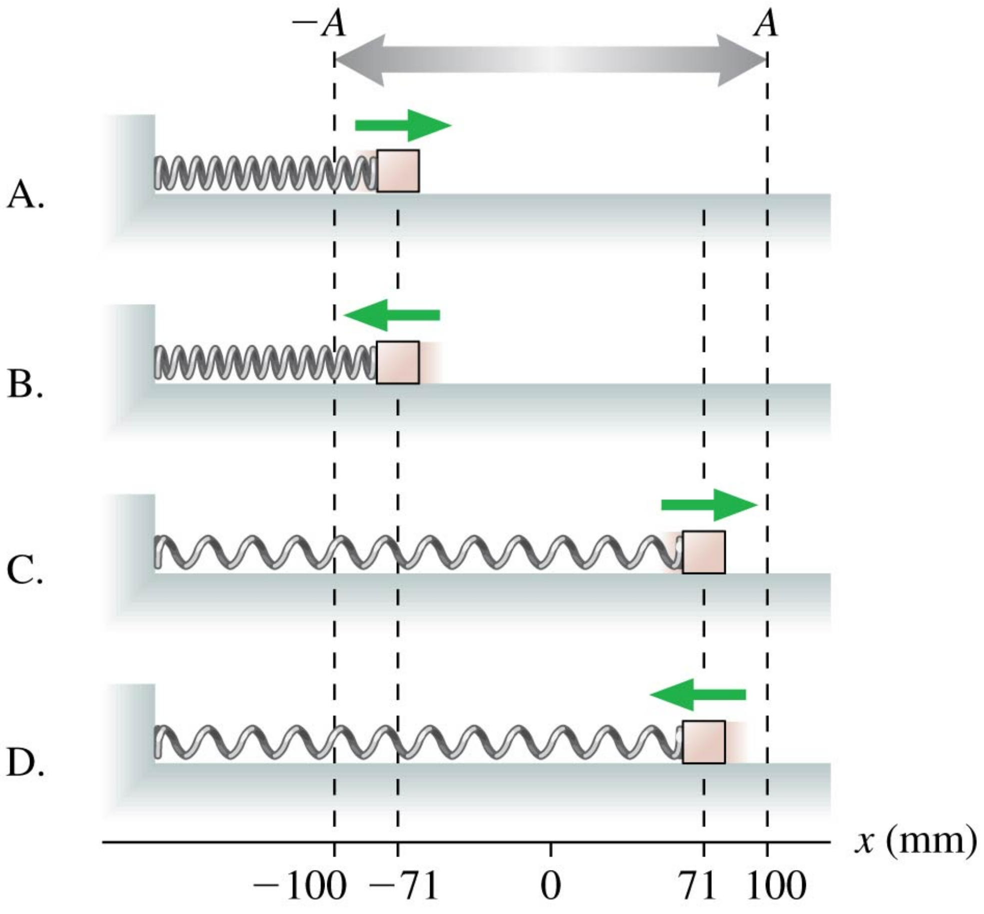
The Phase Constant…three more examples

Oscillations described by the phase constants $\phi_0 = \pi/3, -\pi/3, \pi$ rad



Practice Question:

The figure shows four oscillators at $t = 0$. For which is the phase constant $\phi_0 = -\pi/4$?



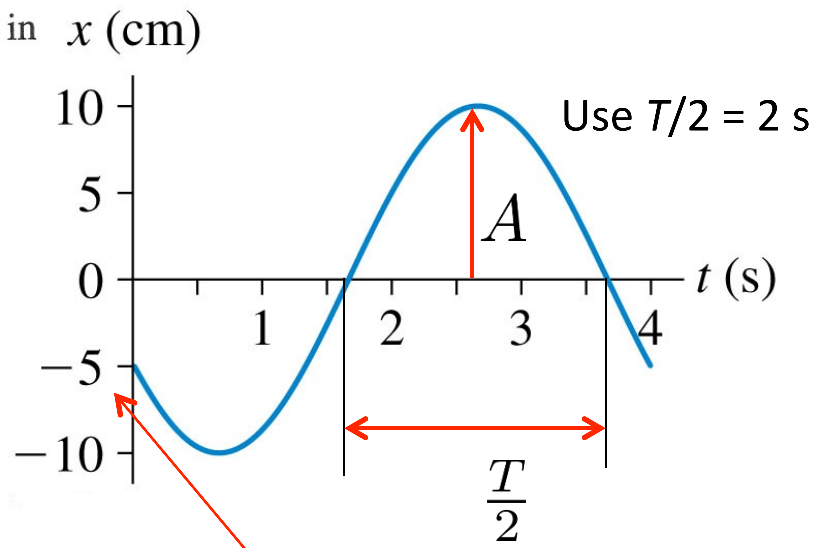
Whiteboard Problem 5: like # 15-7

Shown is the position-versus-time graph of a particle in simple harmonic motion.

a. What is the phase constant?

b. What is the velocity at $t = 0$ s?

c. What is v_{max}?



We can get some info from the graph:

$$A = 10 \text{ cm} \qquad \frac{T}{2} = 2\ s \Rightarrow T = 4\ s$$

$$\text{So, } f = \frac{1}{T} = \frac{1}{4}\ Hz \Rightarrow \omega = 2\pi f = \frac{\pi}{2}\ rad/s$$

$$x(t) = A\cos(\omega t + \phi_0) \qquad v(t) = -A\omega\sin(\omega t + \phi_0)$$

$$\text{a.) Find } \phi_0 : \ \text{Set } t = 0 \quad x(0) = A\cos(\phi_0) = -5\ cm \text{ from the graph}$$

$$\text{So, } \phi_0 = \cos^{-1}\left(\frac{-5}{10}\right) = \cos^{-1}(-.5)$$

What does your calculator say?

$$\text{Actually, } \phi_0 = \pm\frac{2}{3}\pi\ rad\ (\pm 120^\circ)$$

Which one is it? We need a way to figure this out.

WB-5

a) $x(t) = A\cos(\omega t + \phi_0)$

At $t = 0$, $-5 = 10\cos\phi_0 \Rightarrow \cos\phi_0 = -\frac{1}{2}$

$\Rightarrow \phi_0 = \pm 120^\circ$

$\pm 2\pi/3$

$\cos(\theta) = \cos(-\theta)$

$x(t) = 10\cos\left(\omega t + \frac{2\pi}{3}\right)$

$+\frac{2\pi}{3}$ $< \pi$ $\qquad -\frac{2\pi}{3}$

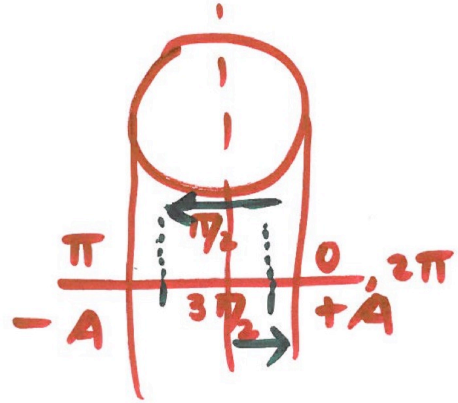
$2\pi - \frac{2\pi}{3} = \frac{4\pi}{3}$

b) $v(t) = -\omega A\sin\left(\omega t + \frac{2\pi}{3}\right)$

$v(t=0) = -\omega A \sin 2\pi/3$ ($A \to 10$)

$\frac{2\pi}{T} = \frac{\pi}{2}$

cm s^{-1}

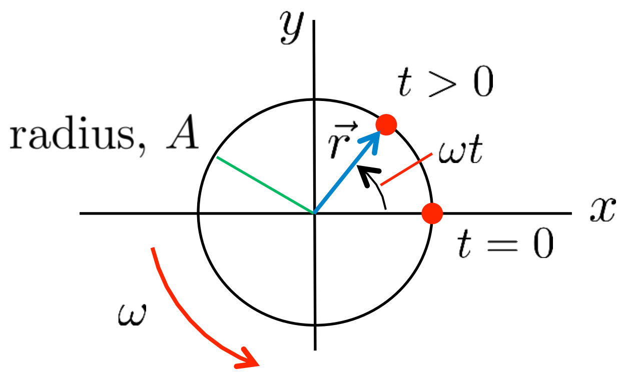


c) $v_{max} = |\omega A| = \frac{\pi}{2}\cdot(10)$

$= 15.5 \text{ cm/s}$

Finding the Phase Constant from the Initial Conditions

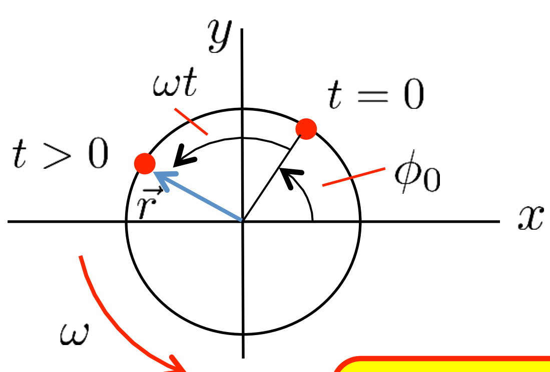
Your author presents a nice way to do this that is a **visualization tool based on uniform circular motion (UCM)**:



The x-component of the object's position vector is:

$$x(t) = A\cos(\omega t) \quad \text{Just SHM with } \phi_0 = 0!$$

What if the object started somewhere else?



Now, the x-component of the object's position vector is:

$$x(t) = A\cos(\omega t + \phi_0) \quad \text{SHM with } \phi_0 \neq 0!$$

So, ϕ_0 is the angle where the equivalent UCM object is at $t = 0$

[Here's a video](#)
(they use the y-axis, but it's the same idea)

Finding the Phase Constant from the Initial Conditions

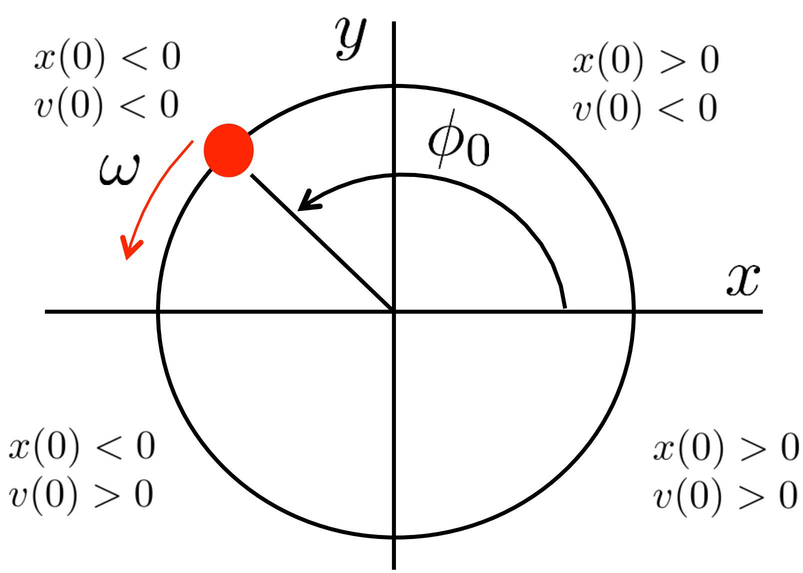
How do we use the UCM tool to find the phase constant?

We know the initial conditions: $x(0)$ and $v(0)$

$$x(t) = A\cos(\omega t + \phi_0) \quad \Rightarrow x(0) = A\cos\phi_0$$

$$v(t) = -A\omega\sin(\omega t + \phi_0) \quad \Rightarrow v(0) = -A\omega\sin\phi_0$$

So, if we know x(0) and v(0), we can find what quadrant the phase constant is in:



Let's see how this works
For Whiteboard problem 5

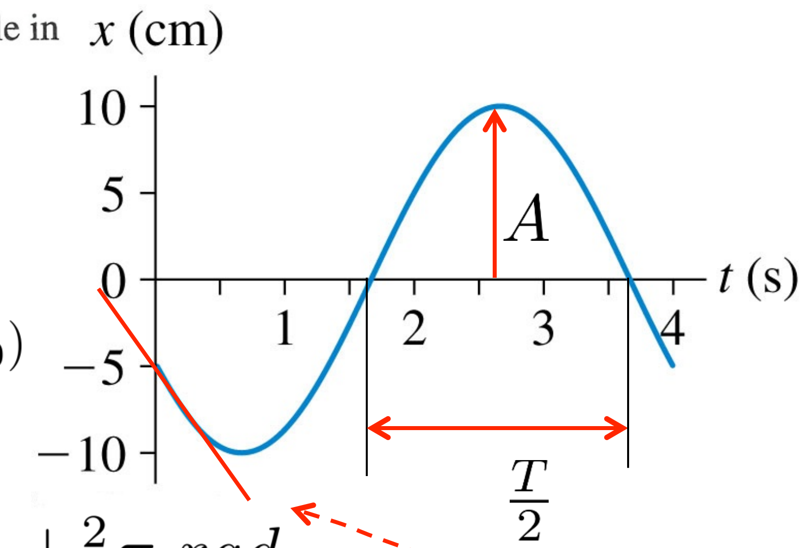
Whiteboard Problem 5: (what we had)

31. | **FIGURE P14.31** is the position-versus-time graph of a particle in simple harmonic motion.

a. What is the phase constant?

b. What is the velocity at $t = 0$ s?

c. What is v_{max}?

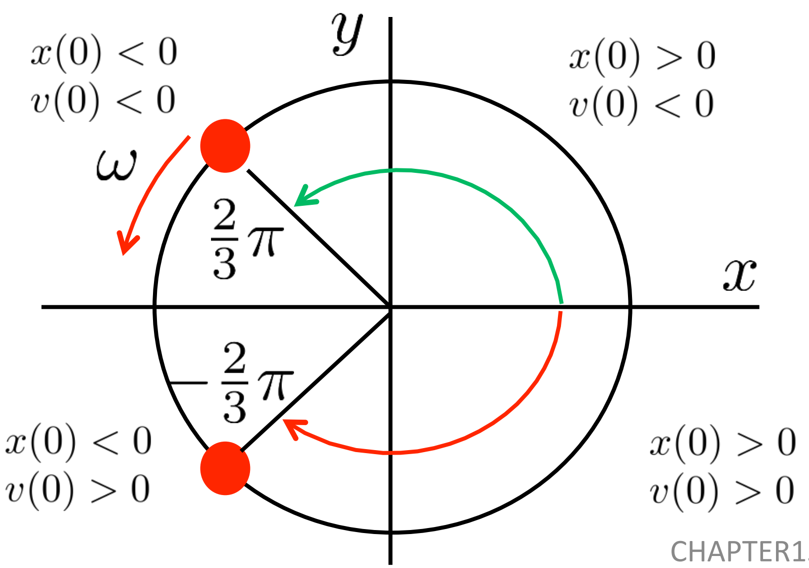


$$x(t) = A\cos(\omega t + \phi_0)$$

$$v(t) = -A\omega\sin(\omega t + \phi_0)$$

a.) Find ϕ_0 :

$$\phi_0 = \cos^{-1}\left(\frac{-5}{10}\right) = \cos^{-1}(-.5) \Rightarrow \phi_0 = \pm\frac{2}{3}\pi\ rad$$



For our problem:

$$x(0) < 0 \text{ and } v(0) < 0$$

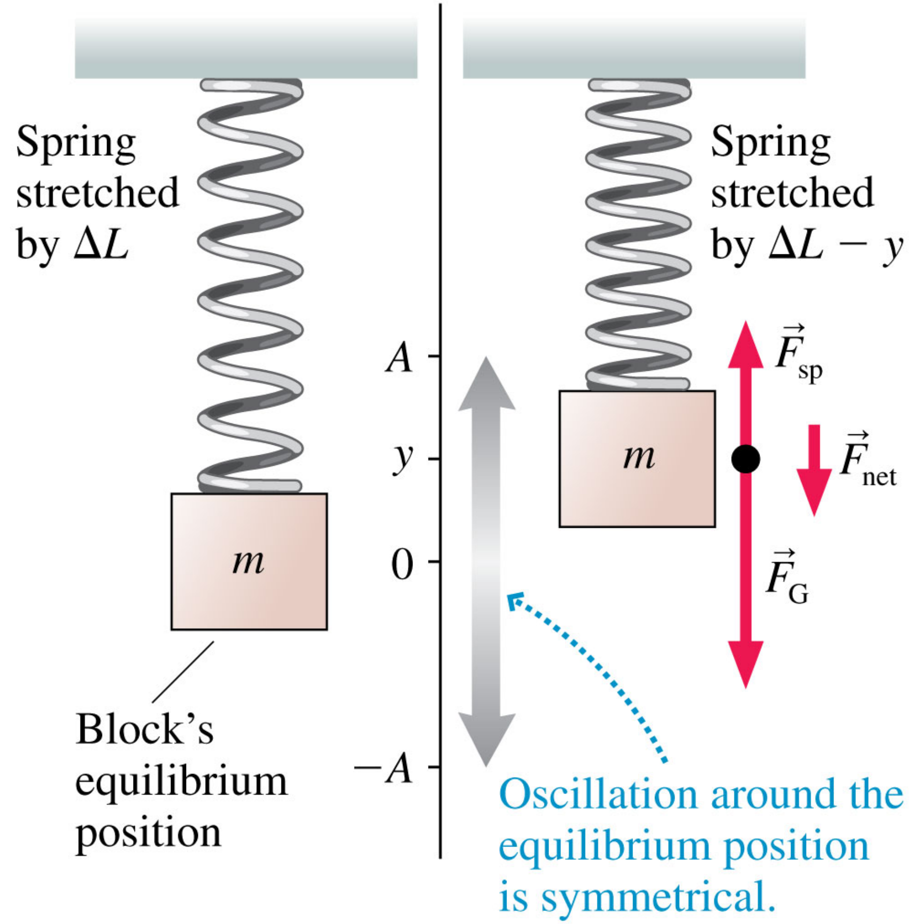
So, ϕ_0 must be in the second quadrant

Thus, $\phi_0 = +\frac{2}{3}\pi$ rad

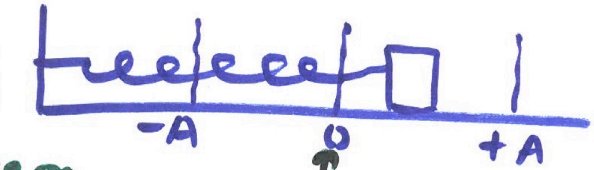
Vertical Oscillations

Motion for a mass hanging from a spring is the same as for horizontal SHM, but the equilibrium position is affected.

$$\Delta L = \frac{mg}{k}$$

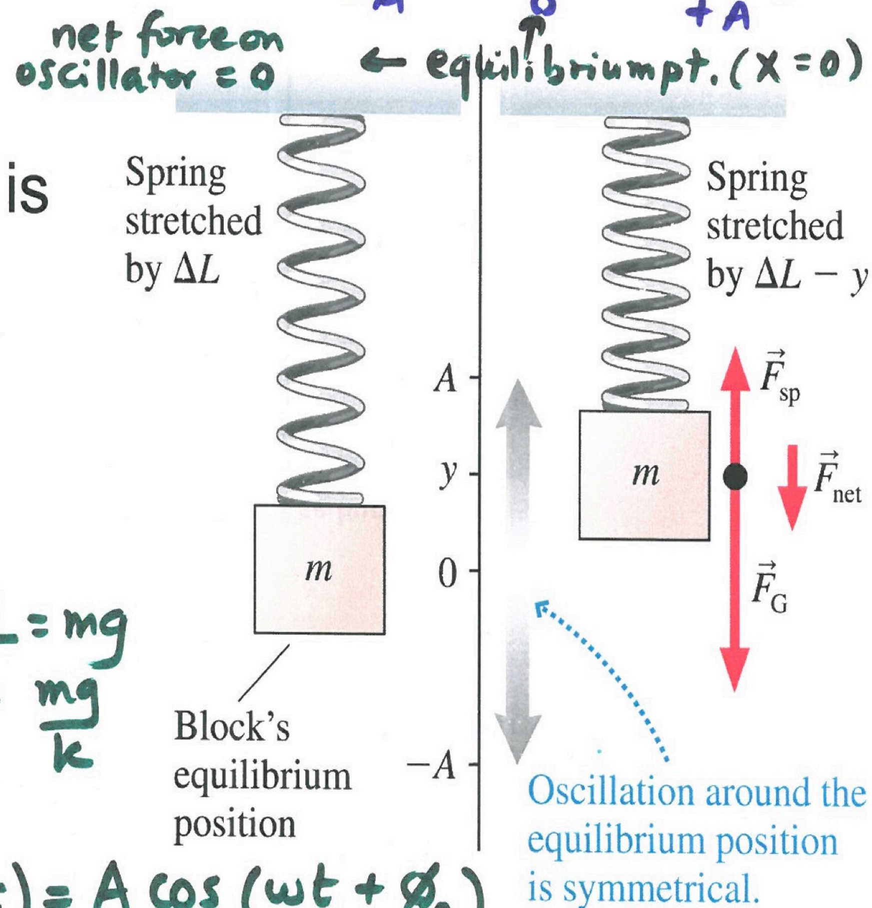


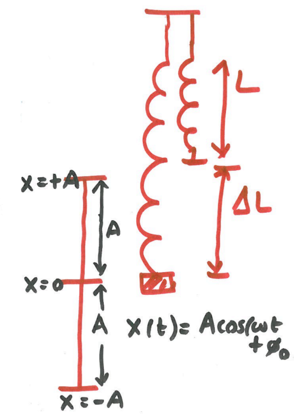
RECALL: IN THE CASE OF HORIZONTAL OSCILLATIONS



Vertical Oscillations

Motion for a mass hanging from a spring is the same as for horizontal SHM, but the equilibrium position is affected.



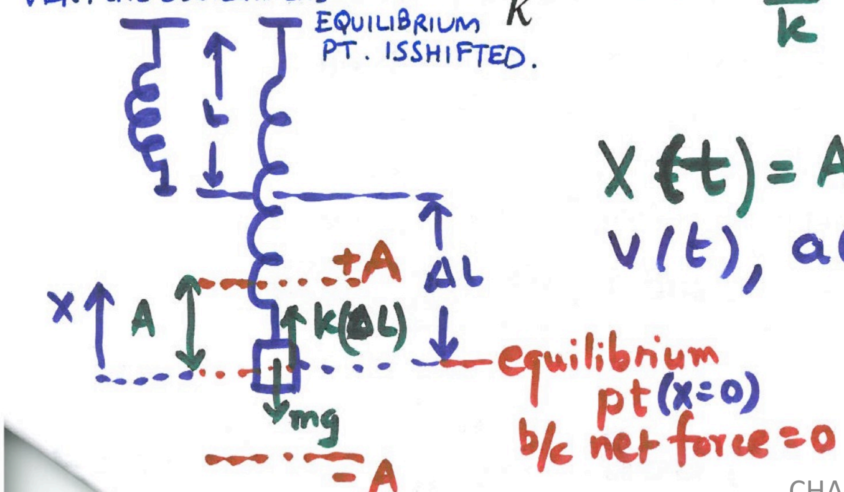


$$\Delta L = \frac{mg}{k}$$

$k\Delta L = mg$

$\Delta L = \frac{mg}{k}$

BUT NOW.... IN THE CASE OF VERTICAL OSCILLATIONS EQUILIBRIUM PT. IS SHIFTED.



$x(t) = A\cos(\omega t + \phi_0)$

$v(t)$, $a(t)$

$\omega = \sqrt{\frac{k}{m}}$ ✓

$E_{TOT} = \frac{1}{2}kA^2 = \frac{1}{2}mv_{max}^2$

$= \frac{1}{2}kx^2 + \frac{1}{2}mv^2$

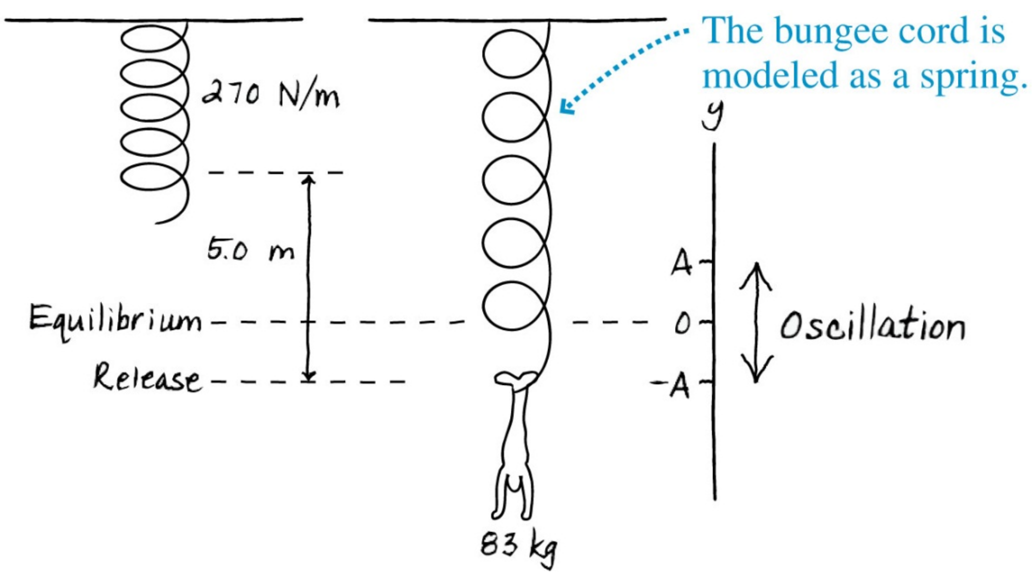
Example 15.6 Bungee Oscillations

Bungee oscillations

An 83 kg student hangs from a bungee cord with spring constant 270 N/m. The student is pulled down to a point where the cord is 5.0 m longer than its unstretched length, then released. Where is the student, and what is his velocity 2.0 s later?

MODEL A bungee cord can be modeled as a spring. Vertical oscillations on the bungee cord are SHM.

VISUALIZE



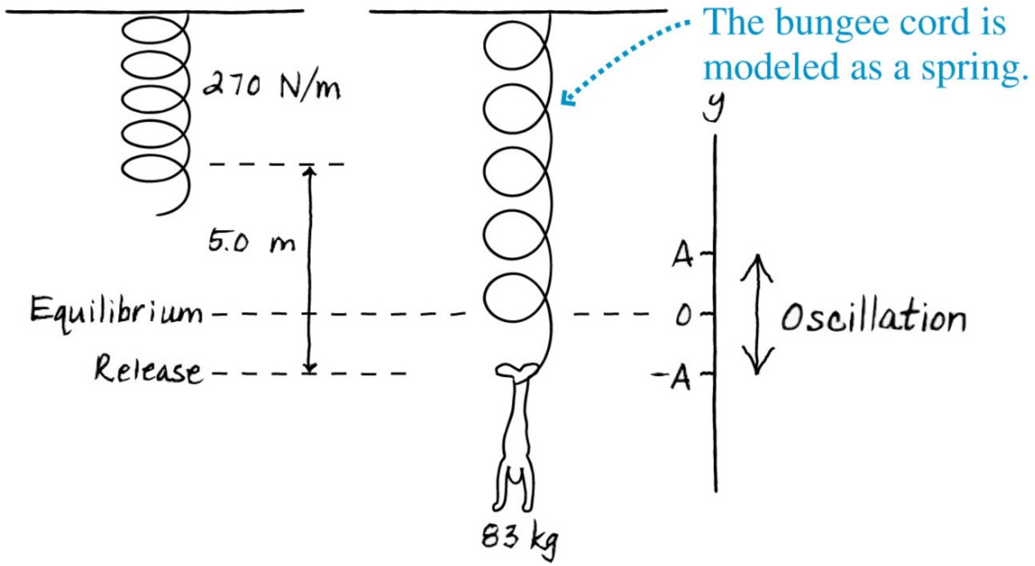
Example 15.6 Bungee Oscillations

Bungee oscillations

SOLVE Although the cord is stretched by 5.0 m when the student is released, this is *not* the amplitude of the oscillation. Oscillations occur around the equilibrium position, so we have to begin by finding the equilibrium point where the student hangs motionless. The cord stretch at equilibrium is given by Equation 14.40:

$$\Delta L = \frac{mg}{k} = 3.0 \text{ m}$$

Stretching the cord 5.0 m pulls the student 2.0 m below the equilibrium point, so $A = 2.0$ m. That is, the student oscillates with amplitude $A = 2.0$ m about a point 3.0 m beneath the bungee cord's original end point.



Example 15.6 Bungee Oscillations

Bungee oscillations

The student's position as a function of time, as measured from the equilibrium position, is

$$y(t) = (2.0\text{ m})\cos(\omega t + \phi_0)$$

where $\omega = \sqrt{k/m} = 1.80$ rad/s. The initial condition

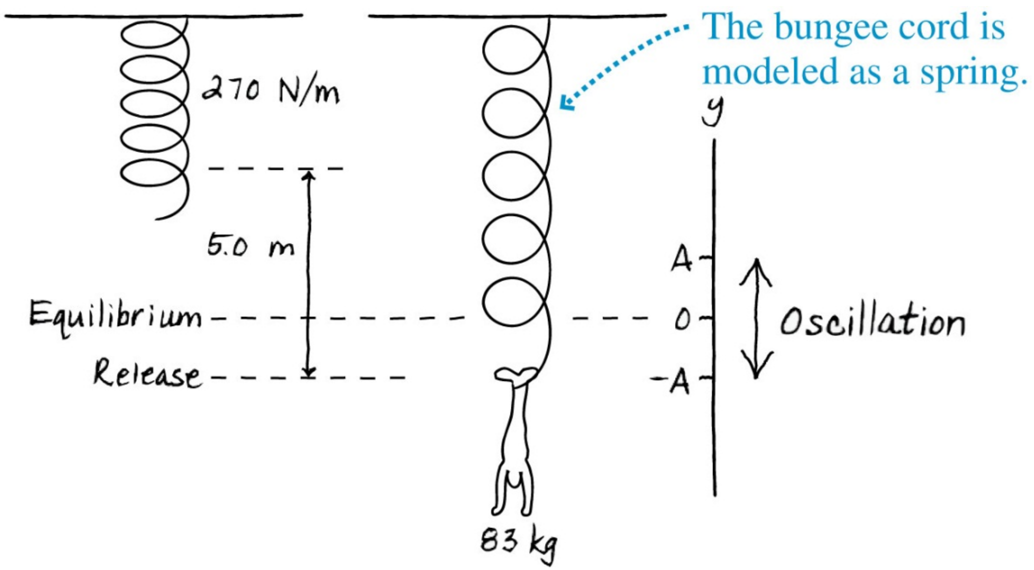
$$y_0 = A\cos\phi_0 = -A$$

requires the phase constant to be $\phi_0 = \pi$ rad. At $t = 2.0$ s the student's position and velocity are

$$y = (2.0\text{ m})\cos\big((1.80\text{ rad/s})(2.0\text{ s}) + \pi\text{ rad}\big) = 1.8\text{ m}$$

$$v_y = -\omega A\sin(\omega t + \phi_0) = -1.6\text{ m/s}$$

The student is 1.8 m *above* the equilibrium position, or 1.2m *below* the original end of the cord. Because his velocity is negative, he's passed through the highest point and is heading down.



Whiteboard Problem 6: #15-64

A penny rides on top of a piston as it undergoes vertical simple harmonic motion with an amplitude of 4.0 cm. If the frequency is low, the penny rides up and down without difficulty. If the frequency is steadily increased, there come a point at which the penny leaves the surface.

a) At what point in the cycle does the penny first lose contact with the piston?
b) What is the maximum frequency for which the penny just barely remains in place for the full cycle?

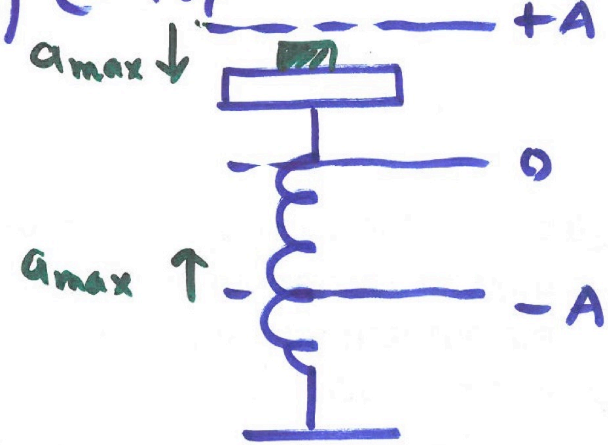
Whiteboard Problem 6: #15-64

A penny rides on top of a piston as it undergoes vertical simple harmonic motion with an amplitude of 4.0 cm. If the frequency is low, the penny rides up and down without difficulty. If the frequency is steadily increased, there come a point at which the penny leaves the surface.

a) At what point in the cycle does the penny first lose contact with the piston?

b) What is the maximum frequency for which the penny just barely remains in place for the full cycle?

a) @ top



$$x(t) = A\cos(\omega t + \phi_0)$$

$$v(t) = -\omega A \sin(\omega t + \phi_0)$$

$$a(t) = -\omega^2 A \cos(\omega t + \phi_0)$$

$$a_{max} = \omega^2 A$$

b) $a_{max} \geq g \rightarrow$ max. accn. ↓ penny has ↓

$$a_{max} = g$$

$$\boxed{\omega^2 A = g} \Rightarrow \omega = \sqrt{g/A} = 15.7 \text{ rad s}^{-1}$$

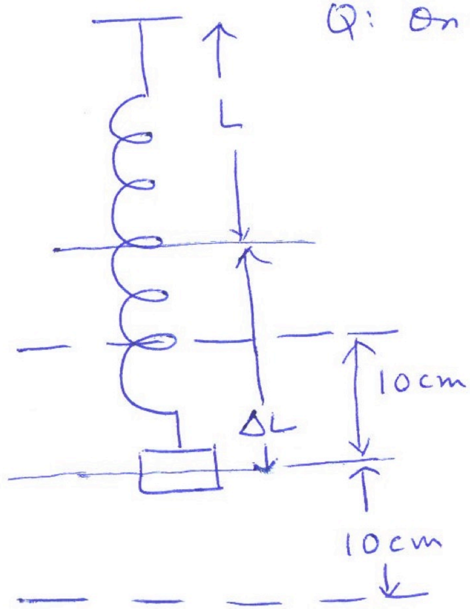
$$\therefore \boxed{f = \frac{\omega}{2\pi} = 2.5 \text{ Hz}}$$

Whiteboard Problem 7: # 15-65

On your first trip to Planet X you happen to take along a 200 g mass, a 40 cm long spring, a meter stick, and a stopwatch. You're curious about the free-fall acceleration g on Planet X, where ordinary tasks seem easier than on Earth, but you can't find this information in your Visitor's Guide. One night you suspend the spring from the ceiling in your room and hang the mass from it. You find that the mass stretches the spring by 31.2 cm. You then pull the mass down 10.0 cm and release it. With the stopwatch you find that 10 oscillations take 14.5 s. Based on this information, what is g?

WB #7

$m = 0.2\text{kg}$; ΔL (see fig. below) $= 31.2\text{cm}$; $A = 10\text{ cm}$; $T = \frac{14.5}{10} = 1.45\text{sec}$.



Q: On planet X, what is 'g'?

$$mg = k\Delta L \Rightarrow g = \frac{k\Delta L}{m} = \frac{k(0.312)}{0.2}$$

$$\Rightarrow g = 1.56k \quad \text{——} \quad (i)$$

NEED TO KNOW k!

But I know $\omega = \sqrt{\frac{k}{m}}$

$$\frac{2\pi}{T} = \frac{2\pi}{1.45}$$

from which I deduce $k = m\omega^2$

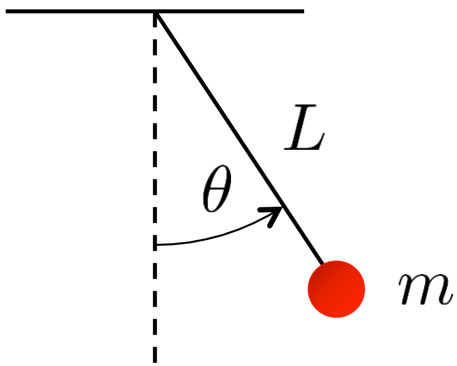
i.e. $k = (0.2)\left(\frac{2\pi}{1.45}\right)^2 = \mathbf{3.76}$ N/m —(ii)

Substitute Eqn (ii) in Eqn. (i) to obtain

$$g = 1.56(3.76) = 5.86\ m/s^2$$

The Simple Pendulum

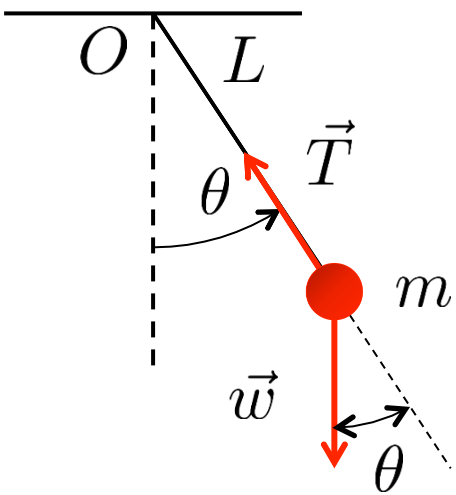
SKIP



A simple pendulum is a point mass on a massless string or rod that can swing around a pivot point.

Your author has one way to show that this is SHM; here's another way:

Free Body Diagram:



$$\circlearrowright + \quad \sum \tau_0 = -wL\sin\theta = I\alpha$$

$$-mgL\sin\theta = (mL^2)\frac{d^2\theta}{dt^2}$$

$$\text{Or, } \frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta$$

This is not SHM, but if we consider only <u>small angles</u>: $\sin\theta \approx \theta$

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta \quad \left[\text{Compare to } \frac{d^2x}{dt^2} = -\frac{k}{m}x, \text{ same DE, same sol'n}\right]$$

<u>So, for the simple pendulum</u>:

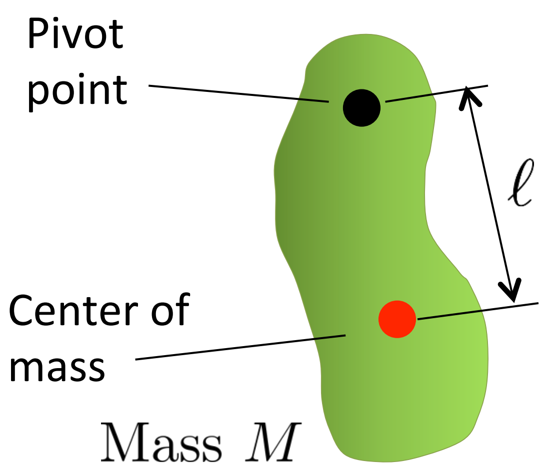
$$\omega = \sqrt{\frac{g}{L}} \quad \text{and } T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}$$

$$\theta(t) = A\cos(\omega t + \phi_0)$$

The Physical Pendulum

SKIP

A physical pendulum is a real object that can rotate about some pivot point:



Your author uses the same steps that we used for a simple pendulum to show that <u>for small angles</u>, this is also SHM where:

$$\omega = \sqrt{\frac{Mg\ell}{I_p}}$$

where: $I_p =$ moment of inertia about the pivot point

$= I_{\text{cm}} + M\ell^2$ (remember the parallel axis theorem?)

Example: HW problem # 57

So, the period is: $T = \dfrac{2\pi}{\omega} = 2\pi\sqrt{\dfrac{I_p}{Mg\ell}}$

Whiteboard Problem 8: # 15-58

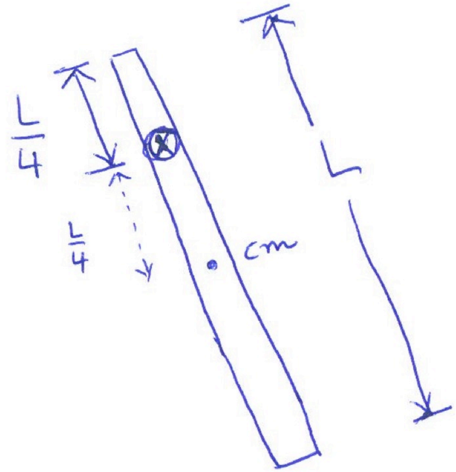
A uniform rod of mass M and length L swings as a pendulum on a pivot at distance $L/4$ from one end of the rod. Find an expression for the frequency f of small-angle oscillations.

Your solution should contain only g, L, and numbers.

SKIP

SKIP

WB #8



$$\omega = 2\pi f = \sqrt{\frac{Mgl}{I_p}}$$

From the figure, $l = \frac{L}{4}$

$$I_p = I_{cm} + Ml^2 = \frac{ML^2}{12} + M\left(\frac{L}{4}\right)^2$$

$$= ML^2\left(\frac{1}{12} + \frac{1}{16}\right)$$

$$= \frac{ML^2}{4}\left(\frac{1}{3} + \frac{1}{4}\right) \rightarrow \frac{7}{12}$$

$$= \frac{7ML^2}{48}$$

$$\therefore f = \frac{1}{2\pi}\sqrt{\frac{Mg\,\frac{L}{4}}{7ML^2} \cdot 48^{12}}$$

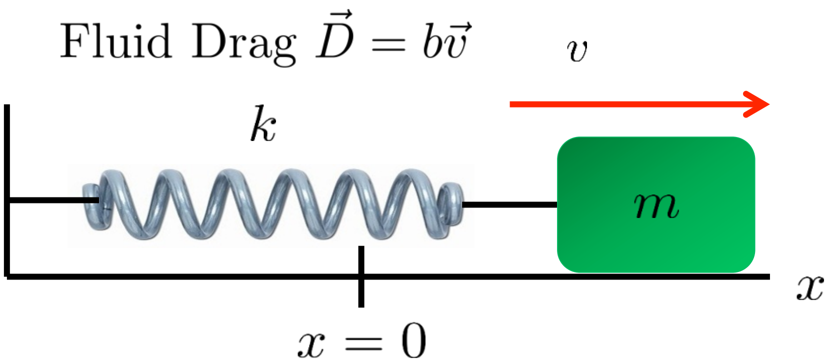
$$f = \frac{1}{2\pi}\sqrt{\frac{12g}{7L}} \quad \text{or} \quad \frac{1}{\pi}\sqrt{\frac{3g}{7L}}$$

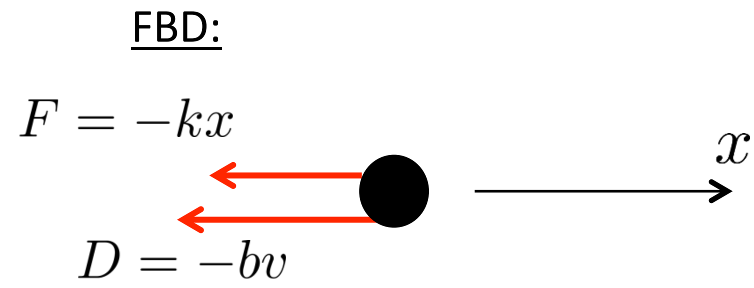
Damped Oscillations

SKIP

In real life, there is always some friction and the amplitude of a SHO will decrease with time. We call this **damping.**

Here's one model of a damped oscillator:





b = damping constant

$$\sum F_x = -kx - bv = ma_x$$

$$\text{or,} \qquad -kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

$$\text{or,} \qquad \frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

This is not such an easy differential equation to guess a solution for; we'll just jump to the solution.

Damped Oscillations – the Solution

The solution for this model of a damped oscillator is:

$$x(t) = \underbrace{A_0 e^{-\frac{t}{2\tau}}}_{A(t)} \underbrace{\cos(\omega t + \phi_0)}_{}$$

decaying part $A(t)$ oscillating part

$$\text{where } \tau = \frac{m}{b} = \textbf{time constant}$$

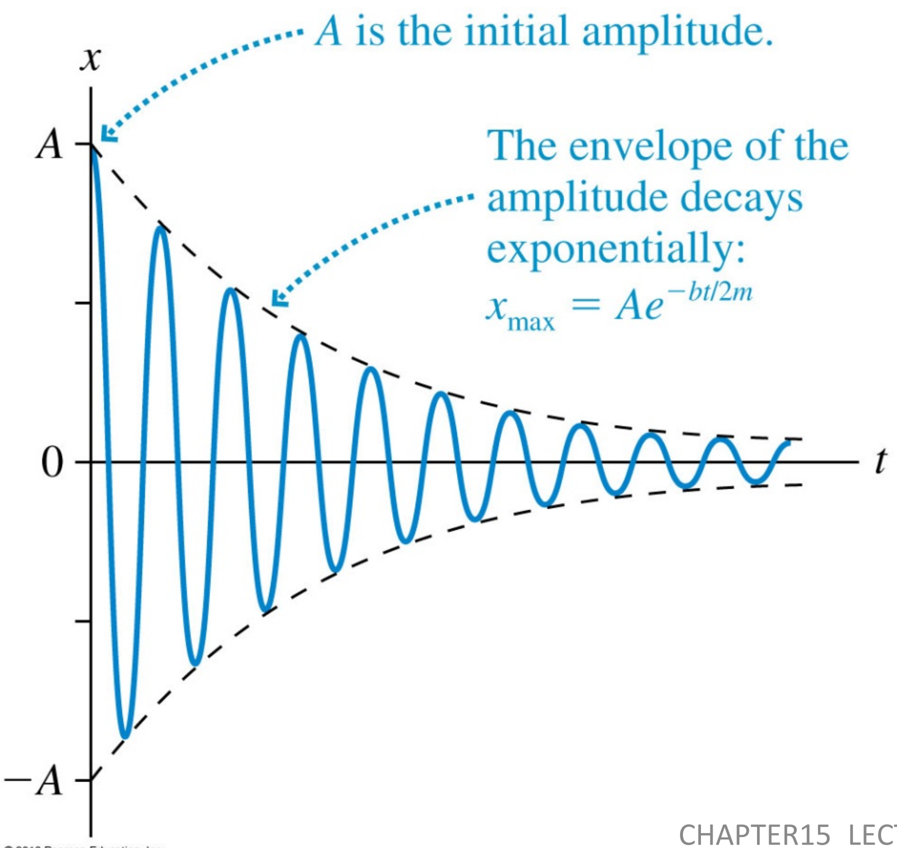
$$\omega = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

$$\omega = \sqrt{\omega_0^2 - \frac{1}{4\tau^2}}$$

$$\approx \omega_0 \ \text{(almost always)}$$

$$\text{Note: } \omega_0 = \sqrt{\frac{k}{m}}$$

The angular frequency of the undamped oscillator.



SKIP

Whiteboard Problem 9: # 14-71

A 250 g air-track glider is attached to a spring with spring constant 4.0 N/m. The damping constant due to air resistance is 0.015 kg/s. The glider is pulled out 20 cm from equilibrium and released. How many oscillations will it make during the time in which the amplitude decays to e^{-1} of its initial value?

Note: the damping constant 0.015 kg/s is the constant b.

Hint: you just have to look at: $A(t) = A_0 e^{-\frac{t}{2\tau}}$

SKIP

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Whiteboard Problem 9: # 14-71

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Say, instead of e^{-1}, this was 20%. Then, set $0.2 = e^{-t/33.33}$ ✓
$t = -33.33 \ln 0.2 = 53.6$ sec

Note: the damping constant 0.015 kg/s is the constant b.

Math tip about exponentials:
If you need to solve the eqn. $a = e^{-t/c}$, where 'a' & 'c' are just #'s, & you need to solve for 't':

Hint: you just have to look at: $A(t) = A_0 e^{-\frac{t}{2\tau}}$

Exploit $\ln e^x = x$!

$\ln a = \ln(e^{-t/c})$
$= -t/c$
$\Rightarrow t = -c \ln a$

- At time 't', say, A(t) is 1/e of A_0. Find 't'.
- Find T, then # of osc. = t/T

$\frac{A_0}{e} = A_0 e^{-1} = A_0 e^{-t/2\tau}$, where $\tau = \frac{m}{b} = \frac{0.25}{0.015} = 16.67$ s

$\therefore A_0 e^{-1} = A_0 e^{-\frac{t}{33.33}}$ $\therefore -1 = -\frac{t}{33.33}$

$\Rightarrow t = 33.33$ s.

$\omega = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} = \sqrt{\frac{4}{0.25} - \frac{0.015^2}{4(0.25^2)}} = \sqrt{16 - 9\times10^{-4}}$
$= 4$ rad s^{-1}

$T = \frac{2\pi}{\omega} = \frac{2\pi}{4} = 1.57$ sec. $\therefore$ # osc $= \frac{33.33}{1.57} = 21.2$ osc.

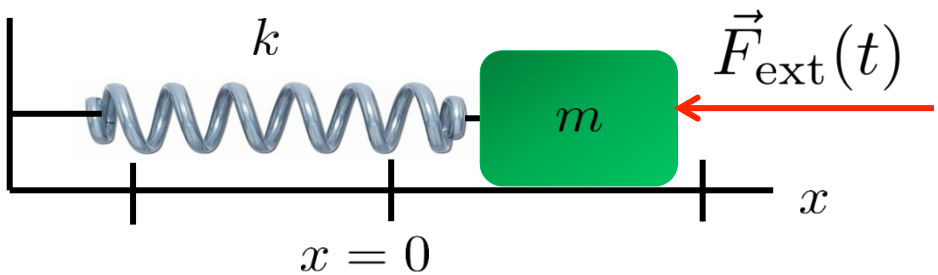
Forced (or Driven) Oscillations and Resonance

- Consider an oscillating system that, when left to itself, oscillates at a **natural frequency** f_0.
- Suppose that this system is subjected to a *periodic* external force of **driving frequency** f.
- The amplitude of oscillations is generally not very high if f differs much from f_0.
- As f gets closer and closer to f_0, the amplitude of the oscillation rises dramatically.

A singer or musical instrument can shatter a crystal goblet by matching the goblet's natural oscillation frequency.

Forced (or Driven) Oscillations and Resonance

What about an applied external force on an oscillator?



For most external forces, this is rather uninteresting, e.g. a constant force just shifts the equilibrium point.

But, a **periodic external force** can do some interesting things:

$$F_{\text{ext}}(t) \propto F_{\text{amp}} \cos(\omega_{\text{ext}} t) \quad (\omega_{\text{ext}} = \text{angular frequency of the applied force})$$

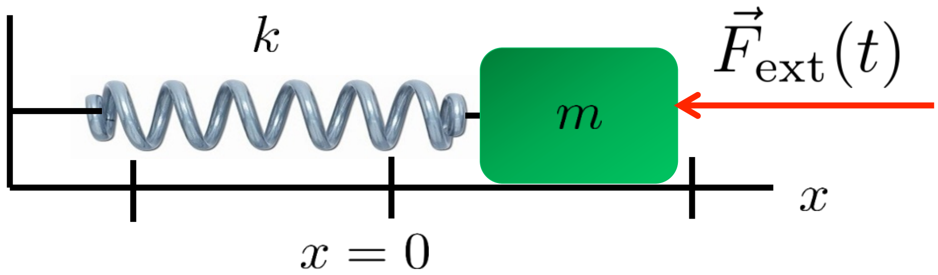
With no damping, the solution is of the form:

$$x(t) \approx \frac{A}{|\omega_0 - \omega_{\text{ext}}|} \cos(\omega_{\text{ext}} t + \phi_0) \quad \omega_0 = \sqrt{\frac{k}{m}} = \text{“natural frequency”}$$

So, for ω_{ext} not near $\omega_0 \Rightarrow$ small amplitude

But, for $\omega_{\text{ext}} \approx \omega_0 \Rightarrow$ Huge amplitude!

Forced Oscillations and RESONANCE



Imagine pushing a child on a swing. Note

1) You apply a periodic force!
2) Timing of your periodic pulses is very important!!

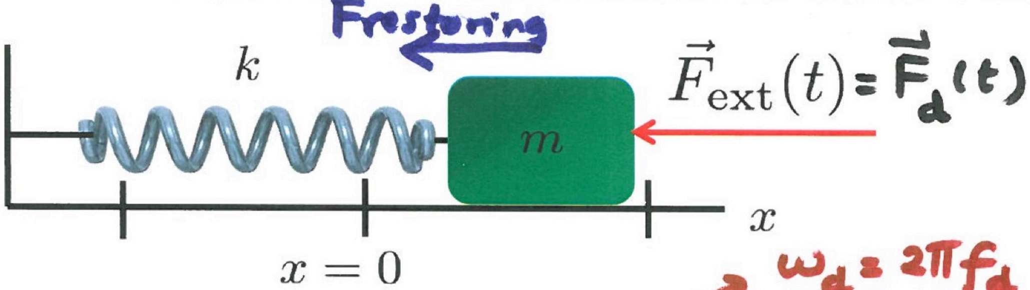
Periodic Force:

Expected response of oscillator:

Note!! There are <u>two</u> frequencies in the problem!
They are the ______________________ and the ___________________________ .

Q: What is the Amplitude of the forced oscillation??
A: Use $\boldsymbol{F}_{\text{net}} = m\boldsymbol{a}$!!

Forced Oscillations and RESONANCE



Imagine pushing a child on a swing. Note

1) You apply a periodic force!
2) Timing of your periodic pulses is very important!!

RESTORING FORCE $= -kx$ $= -kA\cos\omega_d t$

$a = \frac{d^2x}{dt^2} = -\omega_d^2 x$

Periodic Force: $F_d(t) = F_0 \cos\omega_d t$ ($\omega_d = 2\pi f_d$)

Expected response of oscillator: $x(t) = A \cos\omega_d t$ (A?)

Note!! There are two frequencies in the problem!

They are the DRIVING FREQUENCY $\omega_d = 2\pi f_d$ and the NATURAL FREQUENCY $\omega_0 = 2\pi f_0$.

RECALL $\omega_0 = \sqrt{\frac{k}{m}}$

Q: What is the Amplitude of the forced oscillation??

A: Use $F_{net} = ma$!!

$$F_{driving} + F_{restoring} = ma$$

$$\underbrace{F_0 \cos\omega_d t}_{F_{driving}} \underbrace{- kA\cos\omega_d t}_{F_{restoring}} = -m\omega_d^2 A\cos\omega_d t$$

$$\frac{F_0}{m} - \frac{k}{m}A = -\omega_d^2 A \qquad \left(\frac{k}{m} \rightarrow \omega_0^2\right)$$

$$F_0/m = (\omega_0^2 - \omega_d^2)A$$

$$A = \frac{F_0/m}{\omega_0^2 - \omega_d^2}$$

!!!!!

Some Examples of Resonance

A forced oscillator showing resonance – this kid, like any kid, really understands resonance!

Resonant Sound Vibrations

https://www.youtube.com/watch?v=BE827gwnnk4

A famous example of an unplanned resonance – The Tacoma Narrows Bridge

A well-known tragic consequence of resonance: The Challenger shuttle
http://www.spaceflightnow.com/challenger/timeline/