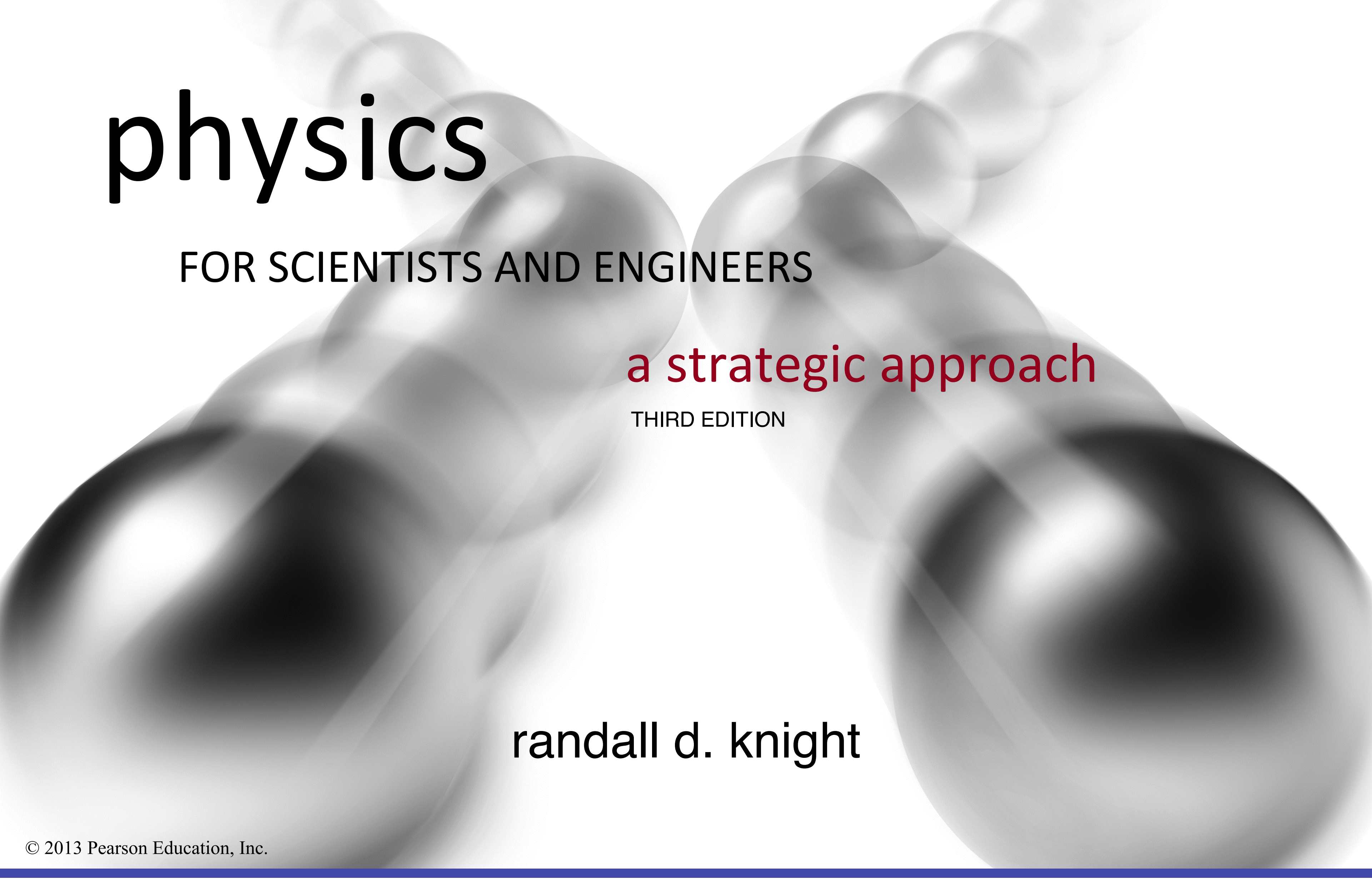
Chapter 15 Lecture – Oscillatory Motion



physics

FOR SCIENTISTS AND ENGINEERS

a strategic approach

THIRD EDITION

randall d. knight

Chapter 15 Oscillations

Chapter Goal: To understand systems that oscillate with *simple harmonic motion.*

Oscillatory Motion

We began Physics 191 with the observation that:

“Everything Moves”

And our goal has been to analyze and understand this motion. **Chapter 12 was devoted to the observation:**

“Some Things Rotate”

Now, in Chapter 14, we want to include the observation that:

“Some Things Wiggle”

Of course, a more technical term than wiggle is ***oscillate****!*

For example, take a meter stick, hold one end firmly clamped to a table, and with your other hand, push the other end down (not far enough to break it), and let it go.

This is an example of damped oscillatory motion which we will get to.

Most of this chapter is devoted to developing a simple model of oscillating systems, **Simple Harmonic Motion**.

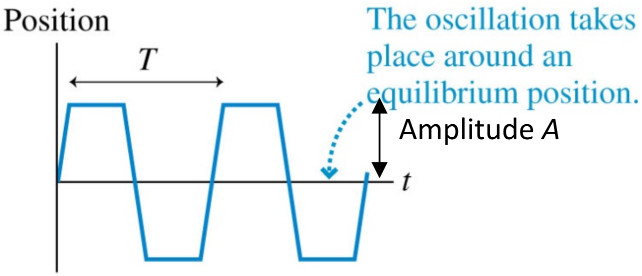
Oscillatory Motion

- Objects that undergo a repetitive motion back and forth around an equilibrium position are called oscillators.
- The time to complete one full cycle, or one oscillation, is called the **period** *T*.
- The number of cycles per second is called the **frequency** *f*, measured in Hz:

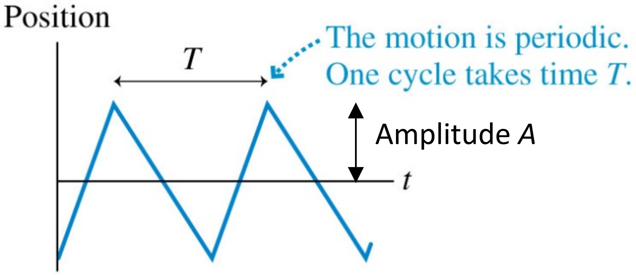
$$f = \frac{1}{T} \qquad \text{or} \qquad T = \frac{1}{f}$$

1 Hz = 1 cycle per second = 1 s^{-1}

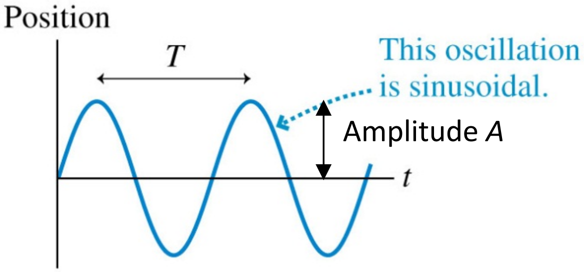
- Maximum displacement from the equilibrium position is called the **Amplitude** *A*.



The oscillation takes place around an equilibrium position.



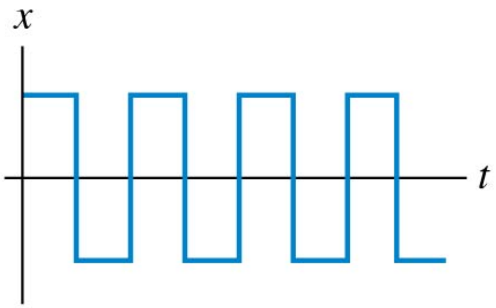
The motion is periodic. One cycle takes time *T*.



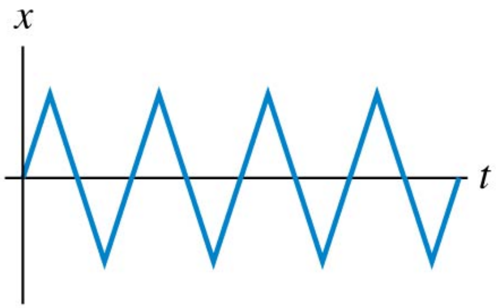
This oscillation is sinusoidal.

Sine-Cosine Oscillation is very, very, very, very Important for Music and Electronic Industry b/c of Fourier Theorem.
Go to Wikipedia “Fourier Series”.

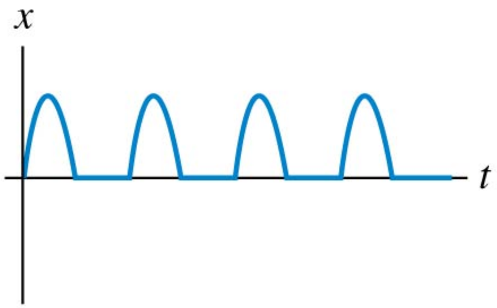
Which oscillation (or oscillations) is SHM?



A.



B.



C.

D. A and B but not C.

E. None are.

Simple Harmonic Motion

Some abbreviations that I use all the time:

SHM = **S**imple **H**armonic **M**otion

SHO = **S**imple **H**armonic **O**scillator

Important Question: In this context, what does **Harmonic** mean?

Mathematically, it means that the motion is describable in terms of **Harmonic Functions** which are **sines** and **cosines**.

In the first three sections, your author introduces SHM from an empirical point of view. He defines things like **period, frequency, angular frequency, amplitude**, the relation to uniform circular motion, and energy considerations.

Read these sections carefully. I'm going to jump ahead to the dynamics treatment of SHM – since we know how to do Newtonian dynamics. After that, we'll come back to some of the basic ideas.

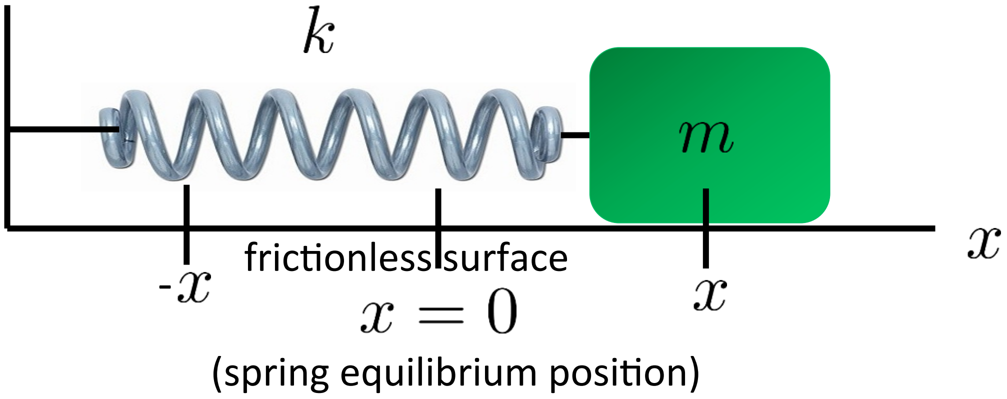
Why sine-cosine oscillations are important…

Fourier Theorem: You can build any arbitrary oscillation by superposing (i.e., adding and subtracting) sine and cosine oscillations of different frequencies. Go to https://en.wikipedia.org/wiki/Fourier_series, and look at:

- Square-wave oscillation
- Triangle-wave oscillation

SHM Dynamics

A useful Model of SHM is a mass on a linear spring. There are many many others, but we have worked with this system and know how to analyze it.



What happens when the mass is released?

It oscillates between +x and –x. What force causes the mass to go back and forth?

Remember Hooke's Law for the Spring:

$$F_x = -kx \Rightarrow \begin{cases} F_x < 0 \text{ for } x > 0 \\ F_x > 0 \text{ for } x < 0 \end{cases}$$

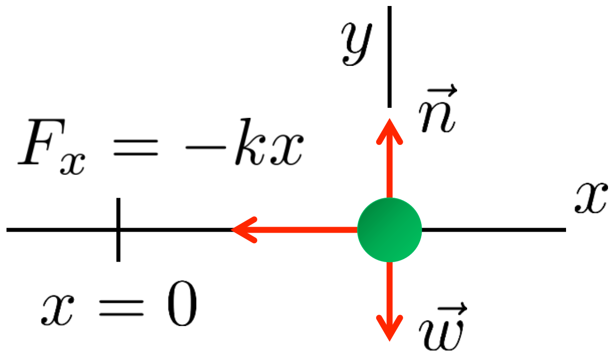
So the spring always pushes or pulls the mass back to x=0, the equilibrium position of the spring.

This is known as a "**Linear Restoring Force**". **Linear** because the force is proportional to the first power of the displacement and **restoring** because the force always returns the system to its equilibrium position.

Important Point: Anytime you have a linear restoring force, you have simple harmonic motion.

SHM Dynamics: Obtaining the Equation

Free Body Diagram for m:



We know how to do this:

$$\sum F_y = n - w = ma_y = 0$$

$$\Rightarrow n = w = mg$$ (Nothing real interesting here.)

$$\sum F_x = -kx = ma_x$$

$$\Rightarrow a_x = -\frac{k}{m}x$$ (Is this constant acceleration?)

So, to describe the motion of the block, **we have to solve:**

$$a_x = \boxed{\frac{d^2x}{dt^2} = -\frac{k}{m}x}$$

(What kind of animal is this?)

This is a second-order linear differential equation for the function x(t).

What do most people do when they're faced with a differential equation?

SHM Dynamics: Solving the Equation

So, our task is to solve the differential equation:

$$\boxed{\frac{d^2x}{dt^2} = -\frac{k}{m}x}$$

First: What is a differential equation?

"A differential equation is a mathematical relation between a function and some of the function's derivatives."

Second: What does it mean to solve a differential equation?

"Solving the differential equation means that you find the function that satisfies the relation."

Third: The mathematically sophisticated way to solve a differential equation is to:

Guess a function!

Fourth: Then you can plug your guess into the differential equation and see if it's correct.

Fifth: If our guess works, how do we know it's the only solution?

Mathematicians have proved the Uniqueness Theorem which says that "linear differential equations have only one unique solution." So if our guess works, it's the only solution.

SHM Dynamics: Solving the Equation

So, for our equation, we are looking for a function x(t) whose second derivative is the negative of the function times some constants.

$$\boxed{\frac{d^2x}{dt^2} = -\frac{k}{m}x}$$

And, we remember from calculus that trigonometric functions have this property, i.e.

$$\text{For } a = \text{constant}, \ \frac{d}{dz}\sin az = a\cos az$$

$$\text{and}, \ \frac{d}{dz}\cos az = -a\sin az$$

So, here is our guess for the function that obeys the differential equation:

$$x(t) = A\cos(\omega t + \phi_0)$$

where: A, ω, and ϕ_0 are constants that we have to determine.

What are these constants trying to tell us? *A* is in m, *w* is in rad/s, f_0 is in rad.

By the way, in our guess, there's a cosine; where's the angle?

There really isn't one. This is an example of using trig functions because of their oscillating nature. There's more to trig functions than right triangles.

SHM: Mass oscillating on a spring

$$a = F/m$$

$$\frac{d^2x(t)}{dt^2} = -\frac{k}{m}\,x(t)$$ ✓

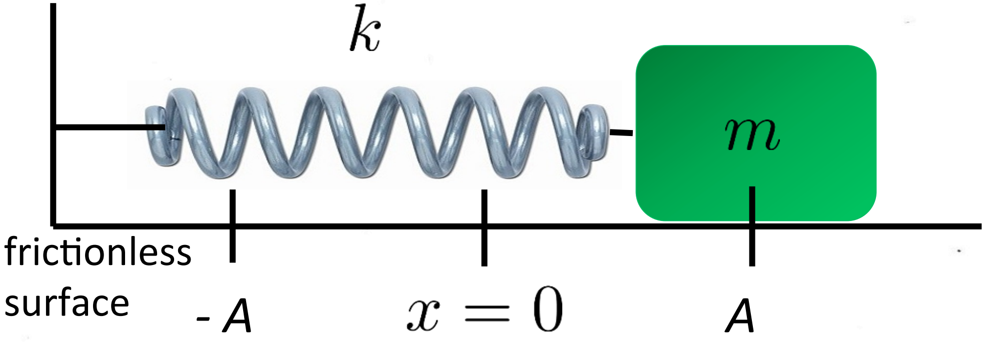
const. that makes 'ωt' have units of radians

$$x(t) = A\cos(\omega t + \phi_0)$$

makes RHS have units of 'm' like LHS

const; PHASE CONSTANT OR INITIAL PHASE

ARGUMENT OF COSINE IS REFERRED TO AS THE "PHASE ANGLE"



SHM: Mass oscillating on a spring

$$a = F/m$$

$$\frac{d^2x(t)}{dt^2} = -\frac{k}{m}\,x(t) \checkmark$$

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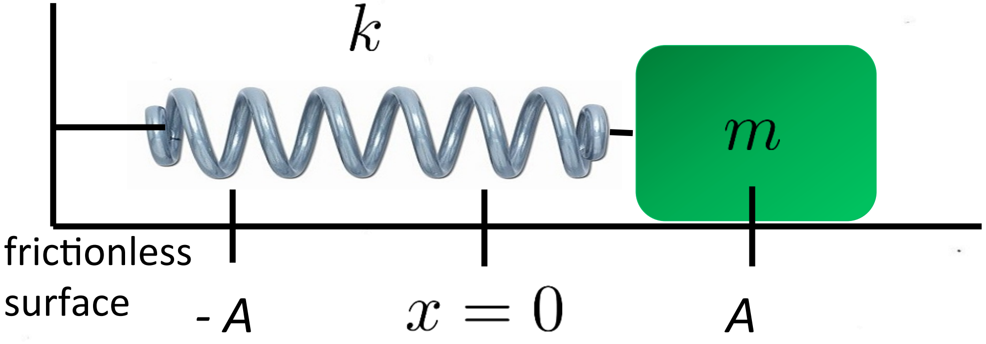


WB Problem 1: What is *w*in SHM of mass-spring?

Show by direct substitution that our guess:

$$x(t) = A\cos(\omega t + \phi_0)$$

is a solution of the differential equation:

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

You should also show what the constant ω is.

For x(t) a solution, must have $\omega = \sqrt{\frac{k}{m}}$

Next up….what is the constant ϕ_0 ???

WB1 Solution

Plonk our <u>guess into 2nd-order</u> DE:

$$\frac{d}{dt}\left[A\cos(\underline{\omega t}+\phi_0)\right] = A(-\sin(\omega t+\phi_0))(\omega)$$

$$= -\omega A\sin(\omega t+\phi_0)$$

$$\frac{d^2}{dt^2}\left[A\cos(\omega t+\phi_0)\right] = \frac{d}{dt}\left[-\omega A\sin(\omega\underline{t}+\phi_0)\right]$$

$$= -\omega^2 A\cos(\omega t+\phi_0) = \text{LHS of D.E.}$$

$$\text{RHS of D.E.} = -\frac{k}{m}A\overset{\shortparallel}{\cos}(\omega t+\phi_0)$$

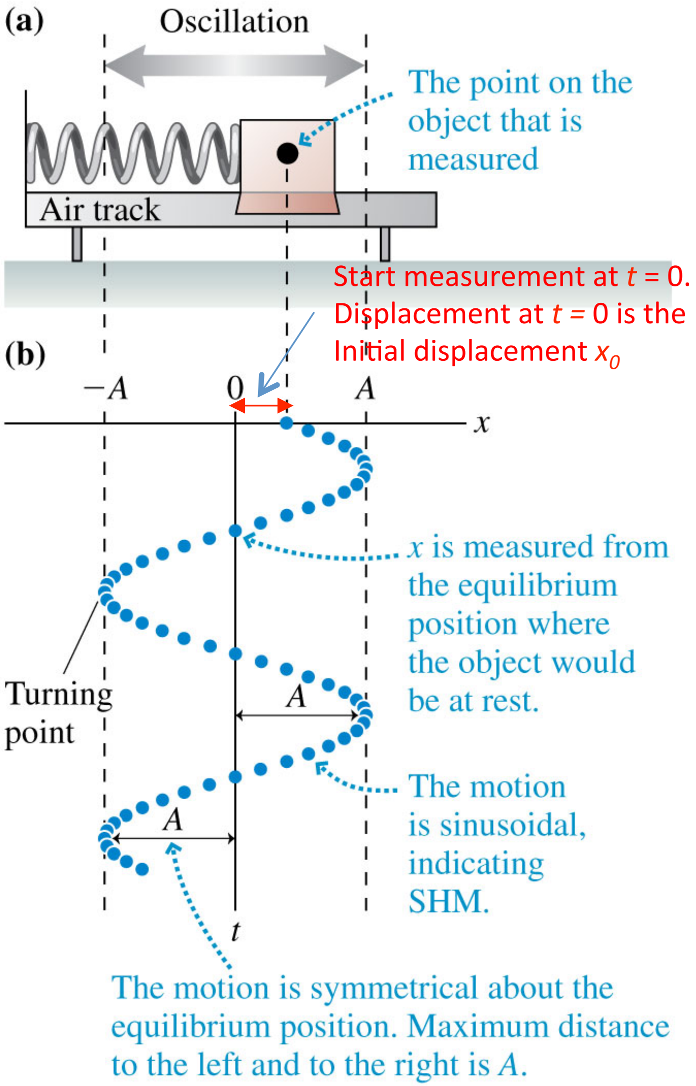
$$\omega^2=\frac{k}{m} \Rightarrow \boxed{\omega=\sqrt{\frac{k}{m}}}$$

SHM – Initial displacement and Phase constant ϕ_0

- Remember, SHM is **simple harmonic motion**.
- In figure (a) an air-track glider is attached to a spring.
- Figure (b) shows the glider’ s position measured 20 times every second.
- SHM: $x(t)=A\cos(\omega t+\phi_0)$

At *t* = 0, $x(t=0)=x_0=A\cos\phi_0$

So...Initial Phase $\phi_0=\cos^{-1}(x_0/A)$



The point on the object that is measured

Start measurement at *t* = 0. Displacement at *t* = 0 is the Initial displacement x_0

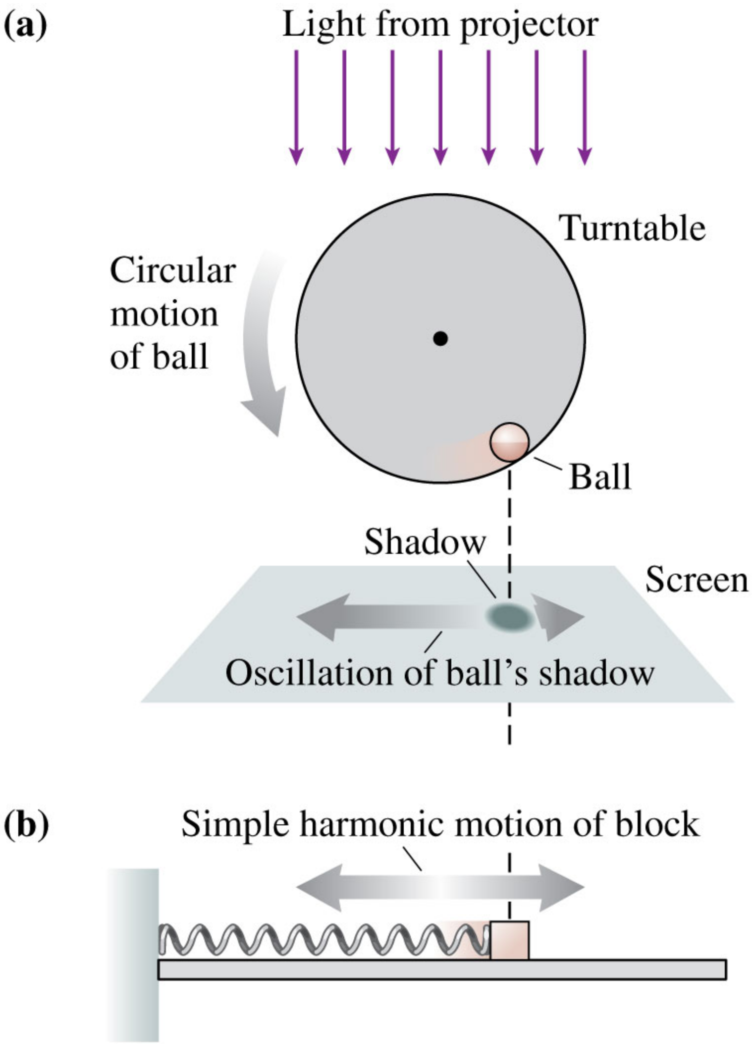
x is measured from the equilibrium position where the object would be at rest.

The motion is sinusoidal, indicating SHM.

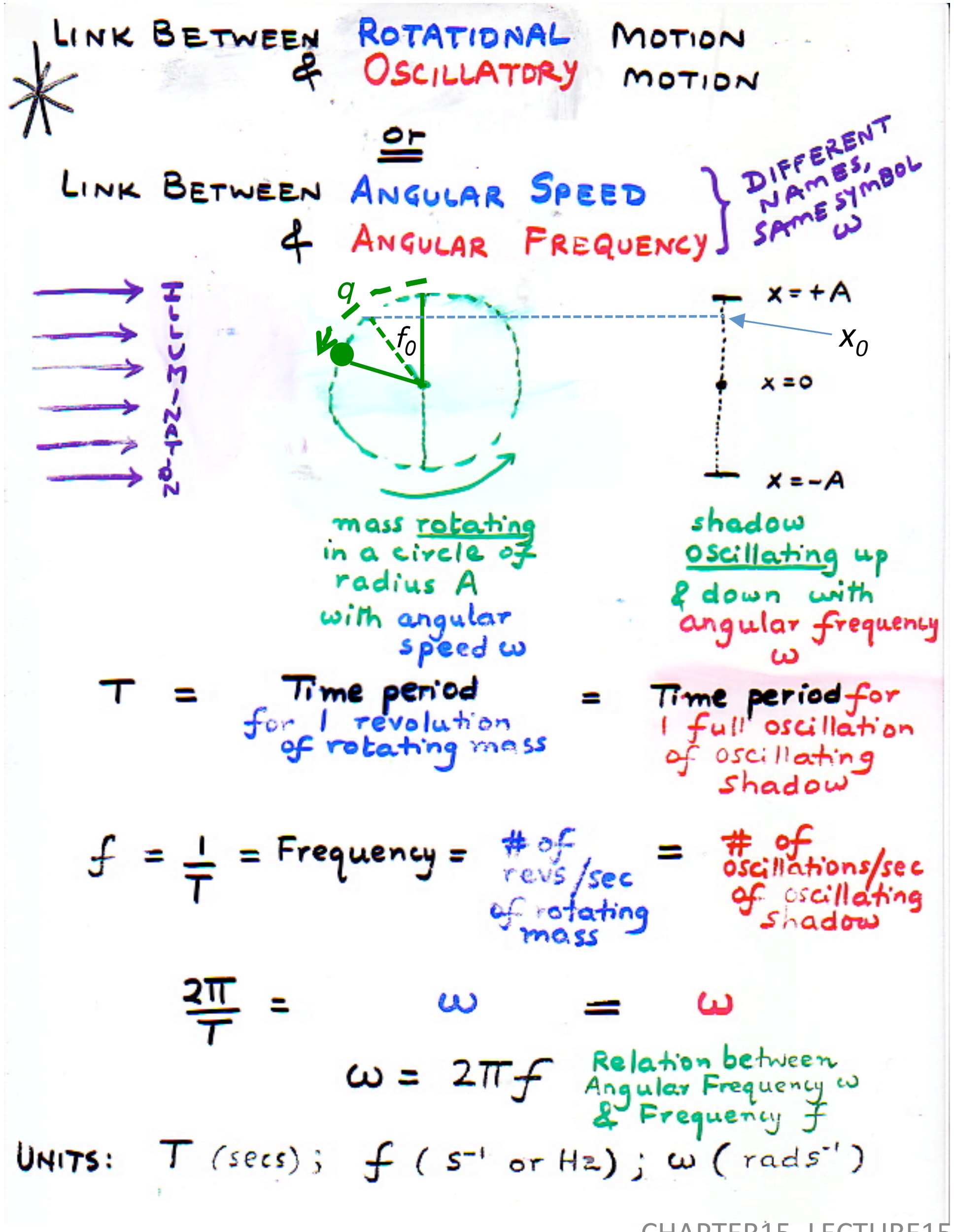
The motion is symmetrical about the equilibrium position. Maximum distance to the left and to the right is *A*.

SHM / Circular Motion: Important insight into ω !!

- Figure (a) shows a "shadow movie" of a ball made by projecting a light past the ball and onto a screen.
- As the ball moves in uniform circular motion, the shadow moves with simple harmonic motion.
- The block on a spring in figure (b) moves with the same motion.



Important insight into ω … continued. And ϕ_0 !



LINK BETWEEN ROTATIONAL MOTION
& OSCILLATORY MOTION

or

LINK BETWEEN ANGULAR SPEED
& ANGULAR FREQUENCY } DIFFERENT NAMES, SAME SYMBOL ω

mass rotating in a circle of radius A with angular speed ω

shadow oscillating up & down with angular frequency ω

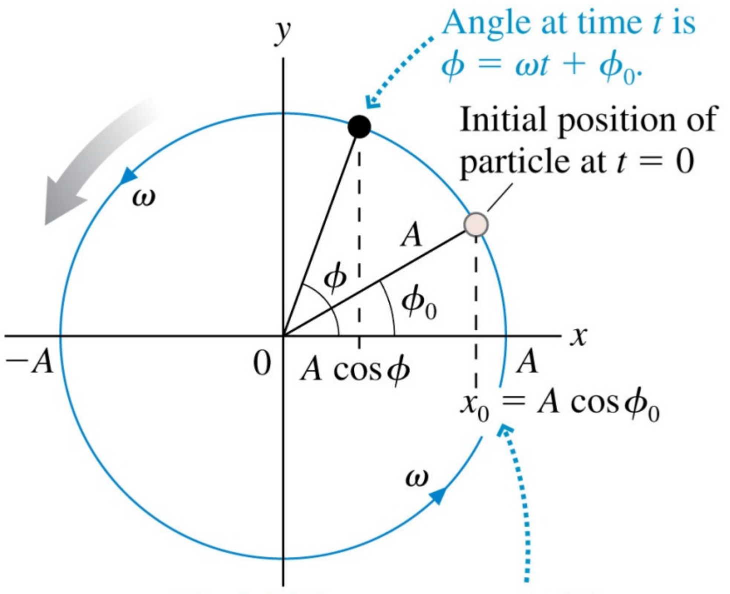
T = Time period for 1 revolution of rotating mass = Time period for 1 full oscillation of oscillating shadow

$f = \frac{1}{T}$ = Frequency = # of revs/sec of rotating mass = # of oscillations/sec of oscillating shadow

$$\frac{2\pi}{T} = \omega = \omega$$

$\omega = 2\pi f$ Relation between Angular Frequency ω & Frequency f

UNITS: T (secs) ; f (s^{-1} or Hz) ; ω ($rads^{-1}$)



Angle at time t is $\phi = \omega t + \phi_0$.

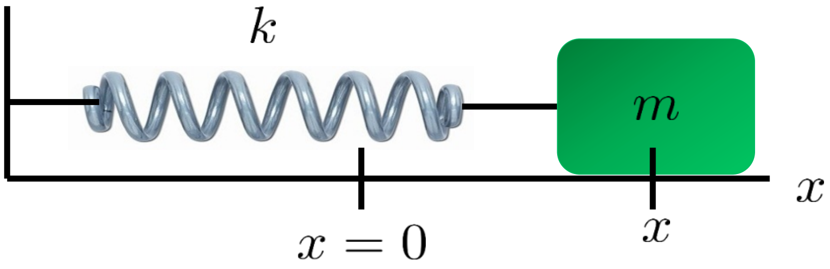
Initial position of particle at $t = 0$

The initial x-component of the particle's position can be anywhere between $-A$ and A, depending on ϕ_0.

Let's watch a video…

SHM: The Basics

The model:



The differential equation:

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

The solution:

$$x(t) = A\cos(\omega t + \phi_0)$$

$$v(t) = \frac{dx}{dt} = -A\omega\sin(\omega t + \phi_0)$$

(Note: be careful using these equations in your calculator. The argument of the sine and cosine is in radians, and your calculator has to be set to radians.)

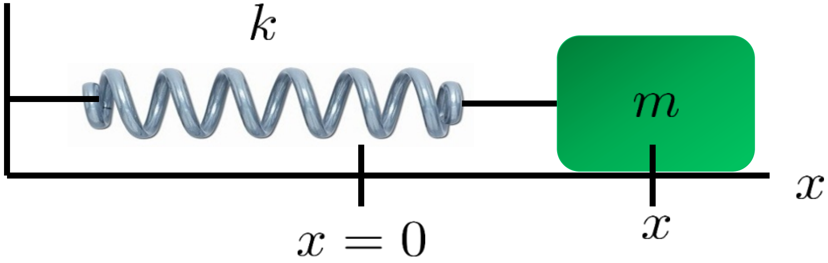
$$A = \text{Amplitude}$$

$$\omega = \sqrt{\frac{k}{m}} = \text{angular frequency } [\tfrac{rad}{s}]$$

$$f = \text{frequency } [\text{Hz} = \text{cycles/second}] = \tfrac{\omega}{2\pi}$$

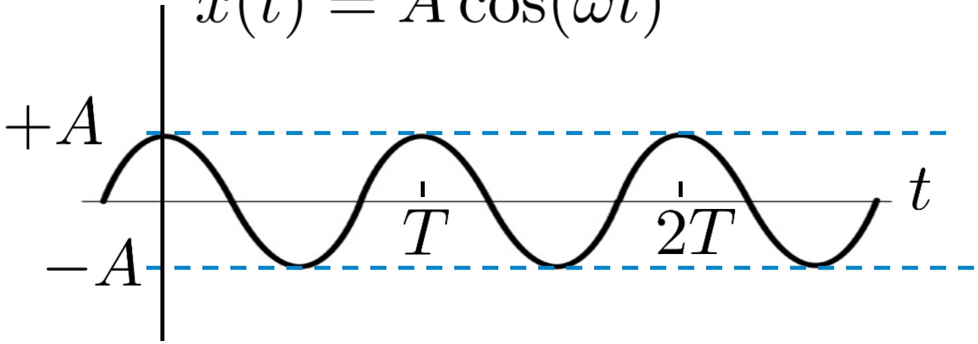
$$T = \text{Period } [s] = \tfrac{1}{f} = \tfrac{2\pi}{\omega}$$

SHM: The Basics



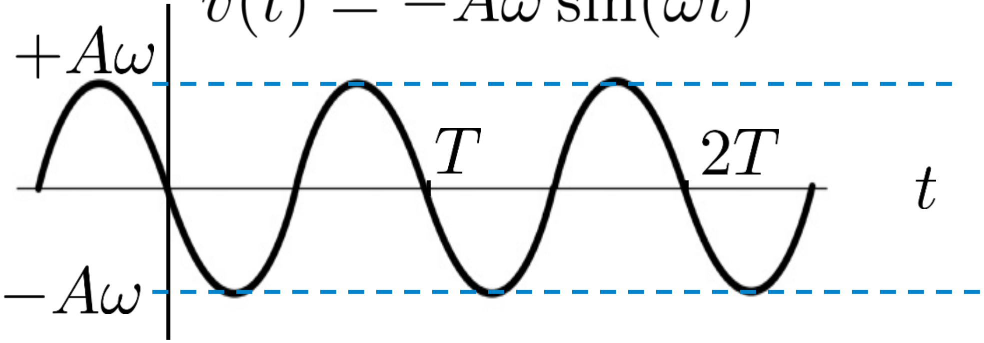
If the block is pulled to $x = +A$ and released from rest at $t = 0$, then $A =$ the amplitude and $\phi_0 = 0$.

$$x(t) = A\cos(\omega t)$$



The position oscillates between $\pm A$ with period T

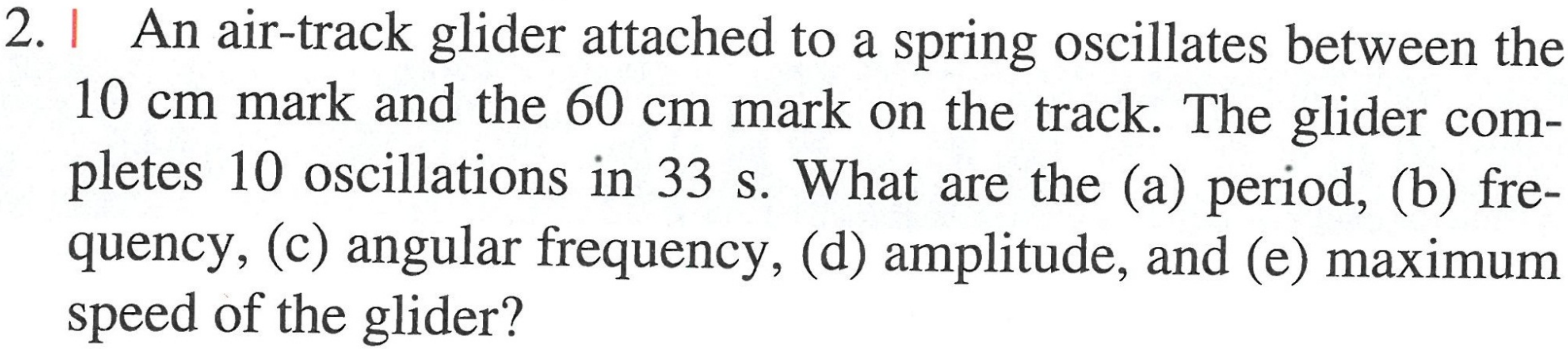
$$v(t) = -A\omega\sin(\omega t)$$



The velocity oscillates between $\pm\omega A$ with period T

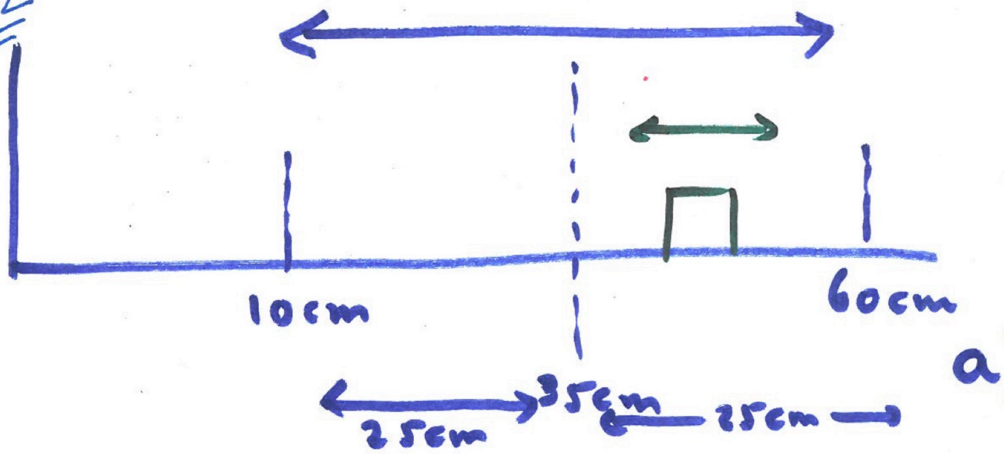
Whiteboard Problem 2: # 15-2

Will help with HW problems 6, 11



2. | An air-track glider attached to a spring oscillates between the 10 cm mark and the 60 cm mark on the track. The glider completes 10 oscillations in 33 s. What are the (a) period, (b) frequency, (c) angular frequency, (d) amplitude, and (e) maximum speed of the glider?

WB2



$$\boxed{\begin{array}{l} A = 25\,cm \\ \quad \text{or } 0.25\,m \end{array}}$$

a) $T = 3.3\,sec.$

$$\frac{33\,sec}{10\,oscillations}$$

b) $f = \frac{1}{T} = \frac{1}{3.3} = 0.303\,Hz$

c) $\omega = 2\pi f = 2\pi(0.303) = 1.904\,rad\,s^{-1}$

d) $A = 0.25\,m$

e) $V_{max} = ?$

$$|V| = |-\omega A \sin(\omega t + \phi_0)|$$

$$|V_{max}| \text{ when } \sin(\omega t + \phi_0) \rightarrow 1$$

$$'' \quad |-\omega A| = \omega A = (1.904)(0.25)$$

$$= 0.476\,m/s$$

Energy and SHM

Since the spring force is a conservative force, the mechanical energy of SHM is conserved:

$$\boxed{E = K + U_s = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \text{constant}}$$

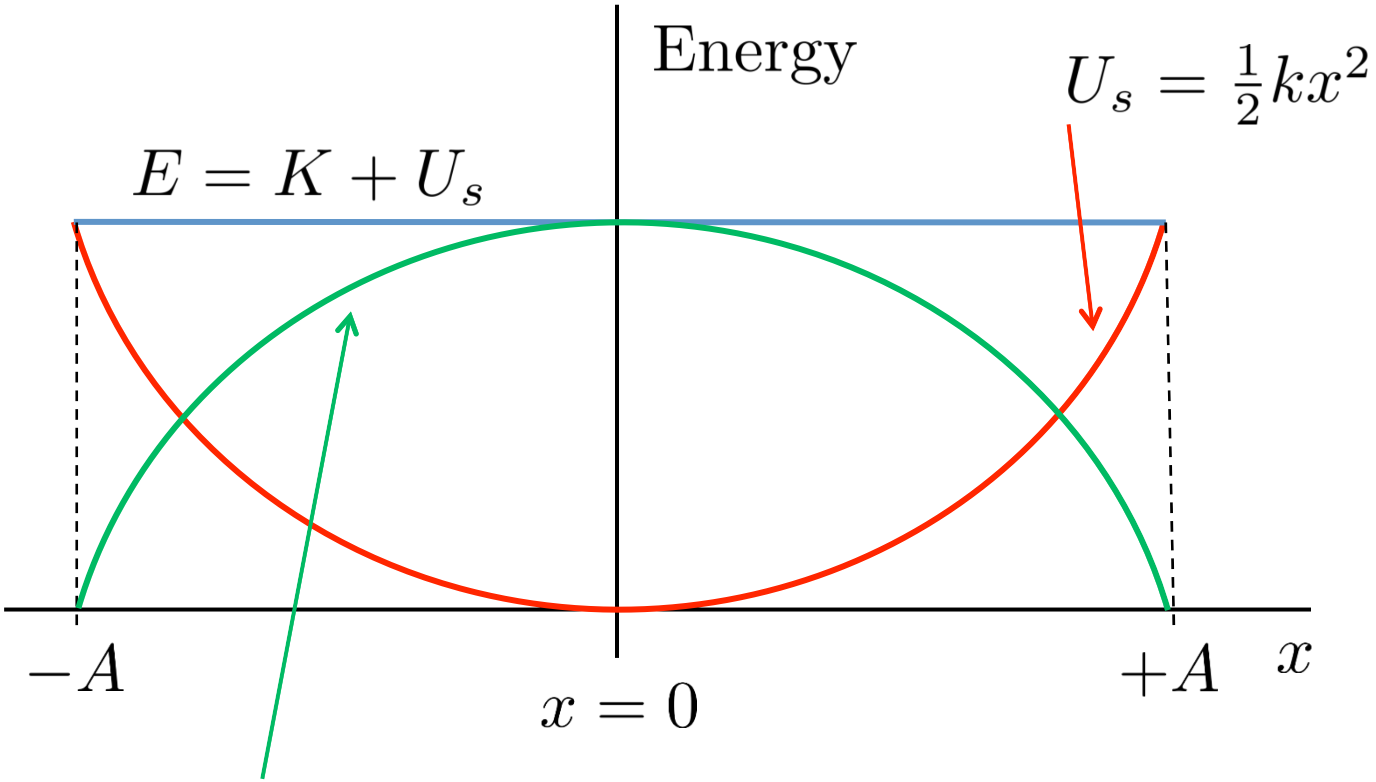
Both the kinetic and potential energies change with time, but their sum is constant. The above expression is for any time. There are two special times that give us two valuable equations for the total energy:

$$\text{At } x = \pm A, v = 0 \quad \Rightarrow \boxed{E = \tfrac{1}{2}kA^2}$$

$$\text{At } x = 0, v = \pm v_{\text{max}} = \pm A\omega \ \Rightarrow \boxed{E = \tfrac{1}{2}mv_{\text{max}}^2 = \tfrac{1}{2}mA^2\omega^2}$$

SHM Energy Plots

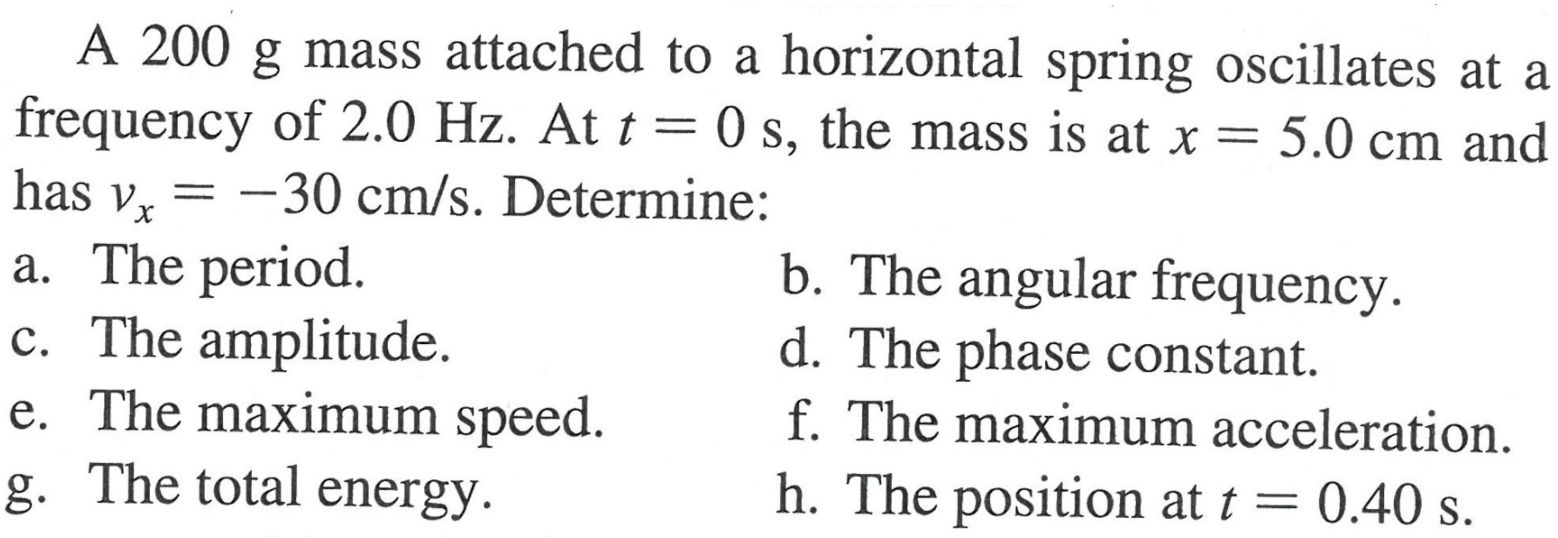
Mechanical energy is conserved in Simple Harmonic Motion



$$K = E - U_s = \frac{1}{2}kA^2 - \frac{1}{2}kx^2 = \frac{1}{2}k(A^2 - x^2)$$

$\pm A$ are called the turning points

Whiteboard Problem 3: # 14-16



A 200 g mass attached to a horizontal spring oscillates at a frequency of 2.0 Hz. At $t = 0$ s, the mass is at $x = 5.0$ cm and has $v_x = -30$ cm/s. Determine:

a. The period.

b. The angular frequency.

c. The amplitude.

d. The phase constant.

e. The maximum speed.

f. The maximum acceleration.

g. The total energy.

h. The position at $t = 0.40$ s.

WB #3 $m = 0.2\,\text{kg};\ f = 2\,\text{Hz}$

a) $T = \frac{1}{f} = 0.5\,\text{s}$

b) $\omega = \frac{2\pi}{T} = 4\pi = 12.57\,\text{rad}\,\text{s}^{-1}$

c) $A = ?$

$$x(t=0) = 0.05 = A\cos(\omega \overset{0}{\cancel{t}} + \phi_0) \Rightarrow 0.05 = \boxed{A}\cos\boxed{\phi_0}$$

$$v(t=0) = -0.3 = -\omega A\sin(\omega \overset{0}{\cancel{t}} + \phi_0) \Rightarrow -0.3 = -12.57\boxed{A}\sin\boxed{\phi_0}$$

2 eqns, 2 unknowns

$$\therefore \boxed{A = 0.0554\,\text{m}} \quad \boxed{A = \frac{0.05}{\cos\phi_0}}\ ;\quad -0.3 = -12.57\,\frac{(0.05)\sin\phi_0}{\cos\phi_0}$$

plonk!

$$\Rightarrow \tan\phi_0 = \frac{0.3}{0.05\,(12.57)} \Rightarrow \boxed{\phi_0 = 0.445\,\text{rads}}$$

d)

e) $$v_{max} = \text{max speed} = \left|-\omega A\,\overset{1,\text{ for max}}{\cancel{\sin(\omega t+\phi_0)}}\right|$$

$$= 12.57(0.0554) = 0.696\,\text{m/s}$$

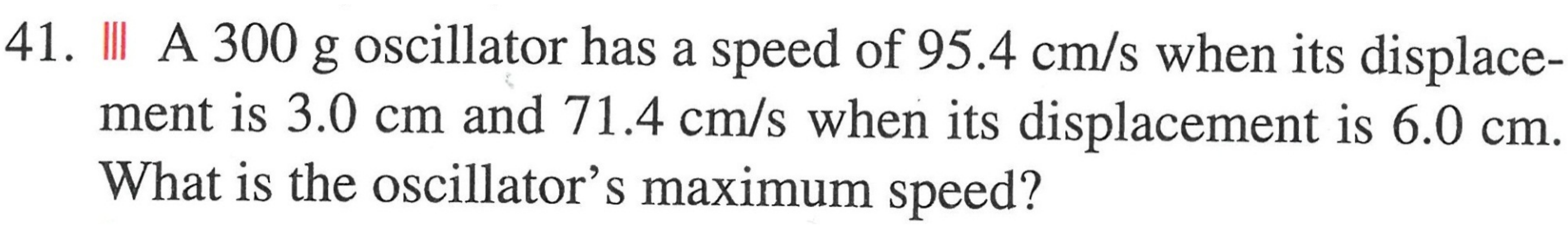
f) $$a_{max} = -\omega^2 A \longrightarrow 12.57^2(0.0554) = 8.75\,\text{m/s}^2$$

$$\text{''}\ -\omega^2 A\,\overset{1,\text{ for max}}{\cancel{\cos(\omega t+\phi_0)}}$$

g) $$E = \frac{1}{2}kx^2 + \frac{1}{2}mv^2 = \frac{1}{2}kA^2 = \boxed{\frac{1}{2}mv_{max}^2} = \frac{1}{2}(0.2)(0.696)^2 = 0.0485\,\text{J}$$

h) $$x(\overset{0.4s}{t}) = A\cos(\omega t + \phi_0) = (0.0554)\cos(12.57\times 0.4 + 0.445) = 0.0372\,\text{m}$$

Whiteboard Problem 4: # 14-41



41. ||| A 300 g oscillator has a speed of 95.4 cm/s when its displacement is 3.0 cm and 71.4 cm/s when its displacement is 6.0 cm. What is the oscillator's maximum speed?

Hint! Note the problem only specifies “speed”, never the velocity.
Which SHM formulas deal with the speed, not velocity?

Whiteboard Problem 4: # 14-41

41. ||| A 300 g oscillator has a speed of 95.4 cm/s when its displacement is 3.0 cm and 71.4 cm/s when its displacement is 6.0 cm. What is the oscillator's maximum speed?

$$E_{TOT} = \boxed{\frac{1}{2}\underline{\underline{k}}x^2 + \frac{1}{2}mv^2} = \frac{1}{2}kA^2 = \boxed{\frac{1}{2}m\underline{\underline{v_{max}^2}}}$$

Hint! Note the problem only specifies "speed", never the velocity. Which SHM formulas deal with the speed, not velocity?

$|v| = 0.954 \text{ m/s} \ @ \ x = 0.03\text{m}$

$|v| = 0.714 \text{ m/s} \ @ \ x = 0.06\text{m}$

First find 'k' by equating E_{TOT} @ 3cm w/ E_{TOT} @ 6cm

$$\boxed{\frac{1}{2}(0.3)(0.954^2) + \frac{1}{2}\underline{\underline{k}}(0.03^2)} = \frac{1}{2}(0.3)(0.714^2) + \frac{1}{2}k(0.06^2)$$

$\Rightarrow k = 44.48 \text{ N/m}$

PLONK INTO EITHER OF E_{TOT} eqns & equate to $\frac{1}{2}mv_{max}^2$

$$\frac{1}{2}(0.3)(0.954^2) + \frac{1}{2}(44.48)(0.03^2) = \frac{1}{2}(0.3)v_{max}^2$$

$\Rightarrow V_{max} = 1.02 \text{ m/s}$