Q: Well, first…what's the rotational analog of mass?

A: Moment of Inertia <u>(Sec. 12.4)</u>

$$I = \text{Moment of Inertia, Units} = [\text{kg m}^2]$$

The moment of inertia of an object is determined by its mass and how the mass is distributed about the axis of rotation.

Key concept: For rotational motion, the spatial distribution of mass about pivot axis matters!

Example: Carrying a heavy long ladder or rod!

<u>Rotational Dynamics (here's the analog for *F* = m*a*):</u>

(Sections 12.6 and 12.7)

A net torque on an object causes an angular acceleration. Or:

$$\alpha = \frac{\tau_{\text{net}}}{I} \quad \text{or} \quad \tau_{\text{net}} = I\alpha$$

This is Newton's 2nd Law for Rotation.

But…what's the definition of Moment of Inertia? How do you calculate it?

PROBLEM-SOLVING STRATEGY 12.1 **Rotational dynamics problems**

(where the rotation is about a fixed axis)

MP

MODEL Model the object as a simple shape.

VISUALIZE Draw a pictorial representation to clarify the situation, define coordinates and symbols, and list known information.

- Identify the axis about which the object rotates.
- Identify forces and determine their distances from the axis. For ~~most~~ All problems it will be useful to draw a free-body diagram.
- Identify any torques caused by the forces and the signs of the torques.

SOLVE The mathematical representation is based on Newton's second law for rotational motion:

$$\boxed{\tau_{\text{net}} = I\alpha \qquad \text{or} \qquad \alpha = \frac{\tau_{\text{net}}}{I}}$$

- Find the moment of inertia in Table 12.2 or, if needed, calculate it as an integral or by using the parallel-axis theorem.
- Use rotational kinematics to find angles and angular velocities.

ASSESS Check that your result has the correct units, is reasonable, and answers the question.

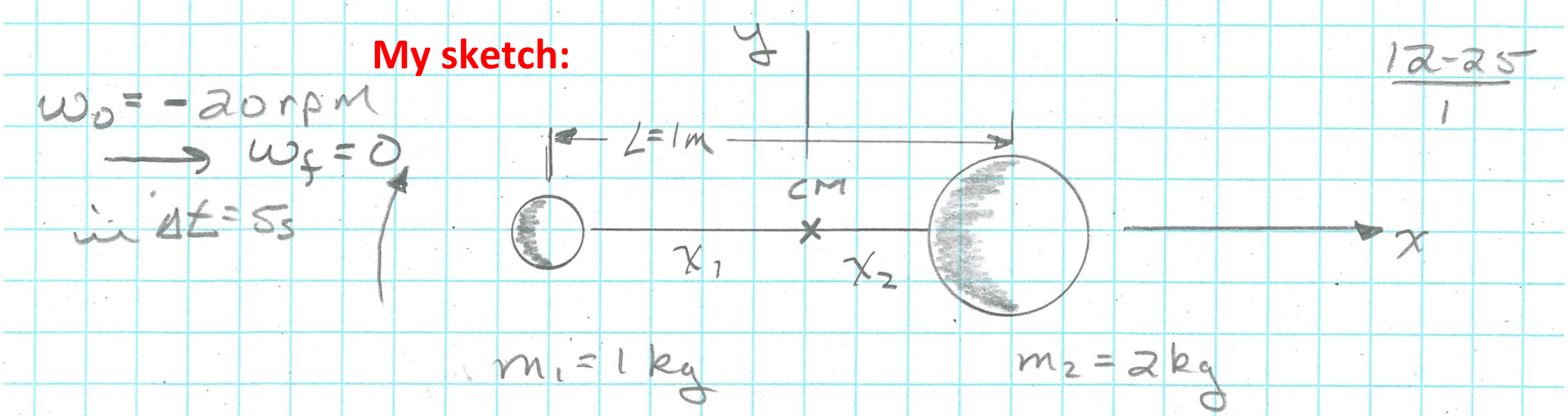
Exercise 26

Whiteboard Problem 7: Problem 12-26

A 1.0 kg ball and a 2.0 kg ball are connected by a 1.0-m-long rigid, massless rod. The rod is rotating cw about its center of mass at 20 rpm. What torque will bring the balls to a halt in 5.0 s?

Steps:
1. **Find the location of the CM**
2. **Find the moment of inertia about the CM**
3. **Find the angular acceleration (assumed constant) from kinematics**
4. **Find the corresponding torque**

My sketch:

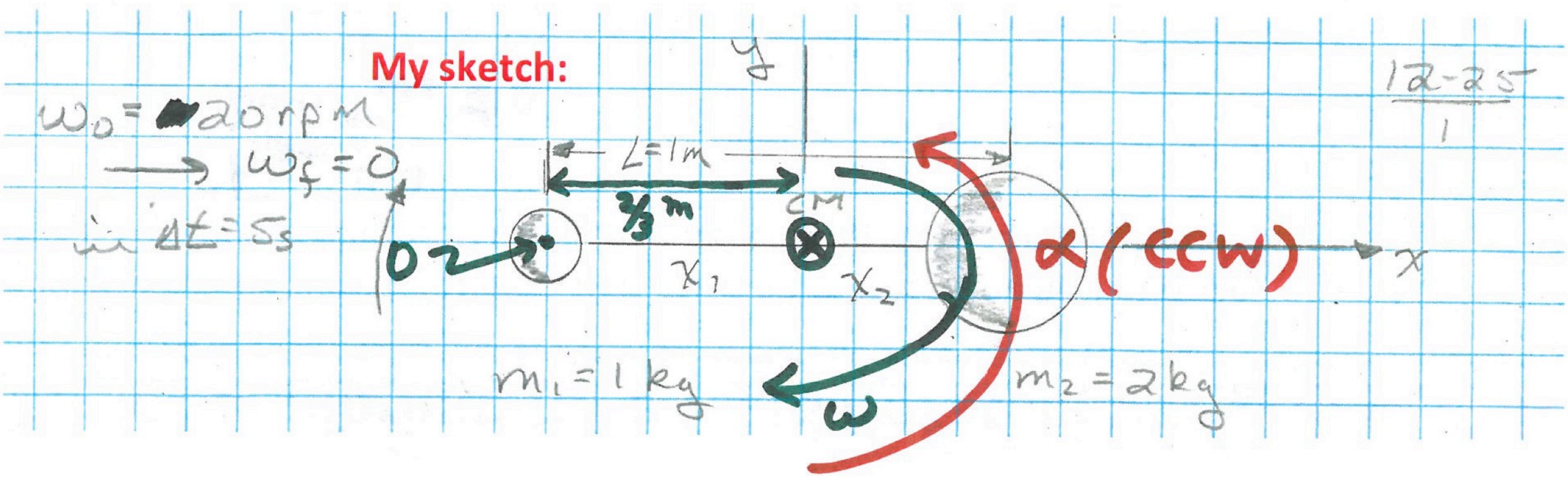


Whiteboard Problem 7: Problem 12-26

A 1.0 kg ball and a 2.0 kg ball are connected by a 1.0-m-long rigid, massless rod. The rod is rotating cw about its center of mass at 20 rpm. What torque will bring the balls to a halt in 5.0 s?

Steps:
1. Find the location of the CM
2. Find the moment of inertia about the CM, I (first locate c.m.)
3. Find the angular acceleration (assumed constant) from kinematics, α
4. Find the corresponding torque $\tau = I\alpha$

My sketch:



- $x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{1(0) + 2(1)}{1+2} = \frac{2}{3}\,m$

- $I_{cm} = m_1 r_1^2 + m_2 r_2^2 = 1\left(\frac{2}{3}\right)^2 + 2\left(\frac{1}{3}\right)^2 = \frac{4}{9} + \frac{2}{9} = \frac{6}{9} = 0.67\ kgm^2$ ✓

- Find α: $\omega_i = \frac{2\cancel{0} \times 2\pi}{3\,\cancel{60}}\ rad\,s^{-1} = \frac{2\pi}{3}\ rad\,s^{-1}$; $\omega_f = 0$; $\Delta t = 5\,sec.$

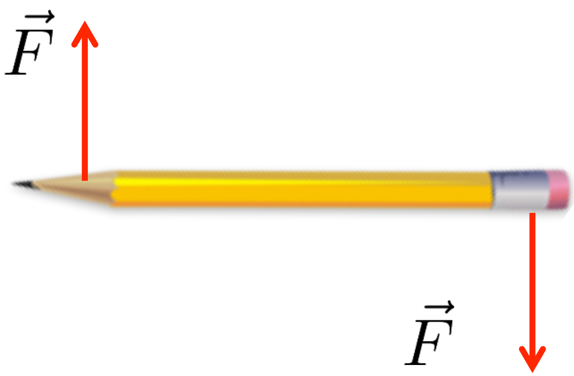
  $\omega_f = \omega_i + \alpha(\Delta t) \Rightarrow \alpha = \frac{\cancel{\omega_f}^{\,0} - \omega_i}{\Delta t} = -\frac{2\pi/3}{5} = -\frac{2\pi}{15}\ rad\,s^{-2}.$

- Use $\tau = I\alpha$

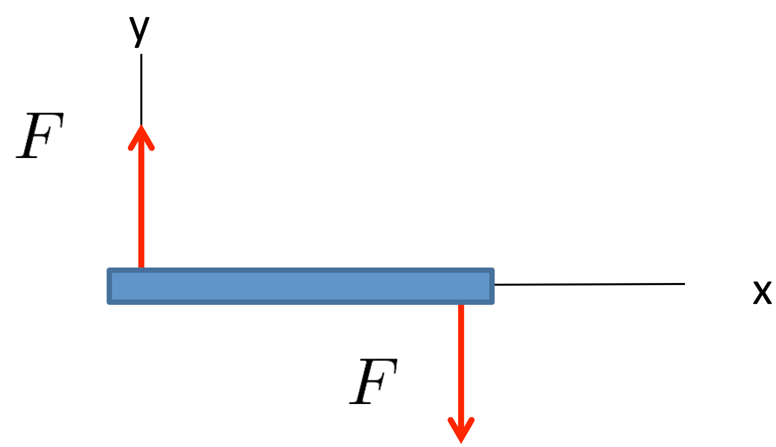
$$\tau = (0.67)\left(\frac{2\pi}{15}\right) = 0.28\,Nm,\ CCW$$

Static Equilibrium for a Rigid Rotating body (Sec. 12.8)

Static Equilibrium for objects that can't be represented as a particle is somewhat different: consider a pencil subject to the two forces:



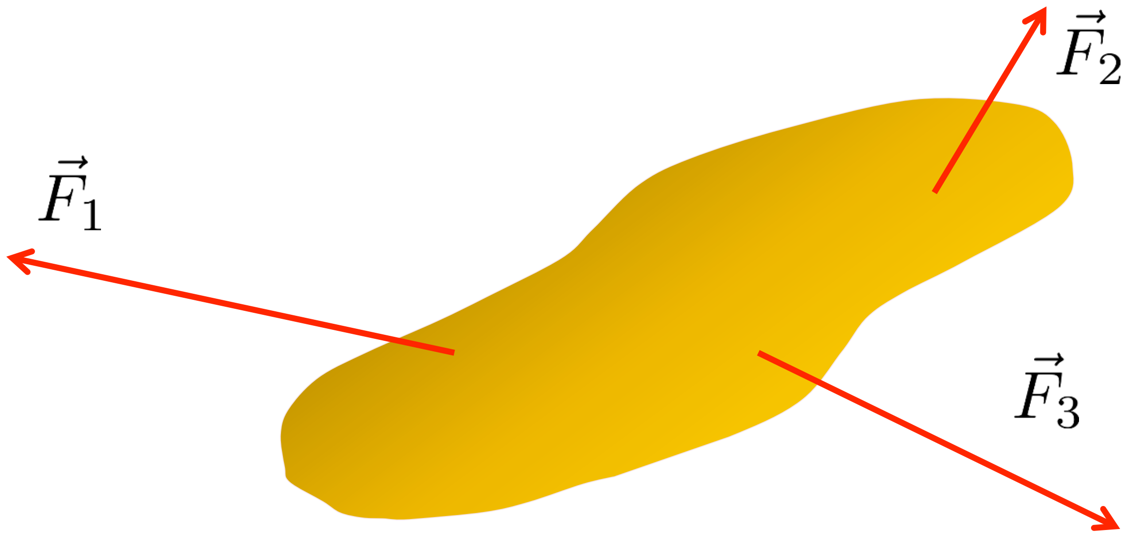
FBD:



$$\sum F_x = 0 \text{ and } \sum F_y = 0, \text{ but is the pencil in equilibrium?}$$

No, it will rotate with an angular acceleration. For real objects in equilibrium, you need the force components to be zero and the sum of the torques about any point to also be zero.

Static Equilibrium* (Sec. 12.8)



The body is in static equilibrium, if:

$$\sum \vec{F}_{\text{net}} = 0$$

$$\tau_{\text{net}} = 0$$

Engineering majors will take an entire course in static equilibrium:* **MME 211, Static Modeling of Mechanical Systems.

Use in Component Form:

$$\sum F_x = 0 \qquad \sum F_y = 0$$

$$\sum F_z = 0 \;\; \text{(if necessary)}$$

$$\sum \tau(\text{from all } \vec{F}_i) = 0$$

The sum of the torques about **ANY POINT** must be zero!

PROBLEM-SOLVING STRATEGY 12.2

Static equilibrium problems

MP

MODEL Model the object as a simple shape. (not a point)

VISUALIZE Draw a pictorial representation showing all forces and distances. List known information. (and a free body diagram!) The weight of an object acts through its CM

- Pick any point you wish as a pivot point. The net torque about this point is zero. (i.e. choose any point to sum torques about)
- Determine the moment arms of all forces about this pivot point.
- Determine the sign of each torque about this pivot point. (Usually, use CCW as +)

SOLVE The mathematical representation is based on the fact that an object in total equilibrium has no net force and no net torque:

$$\vec{F}_{net} = \vec{0} \qquad \text{and} \qquad \tau_{net} = 0$$

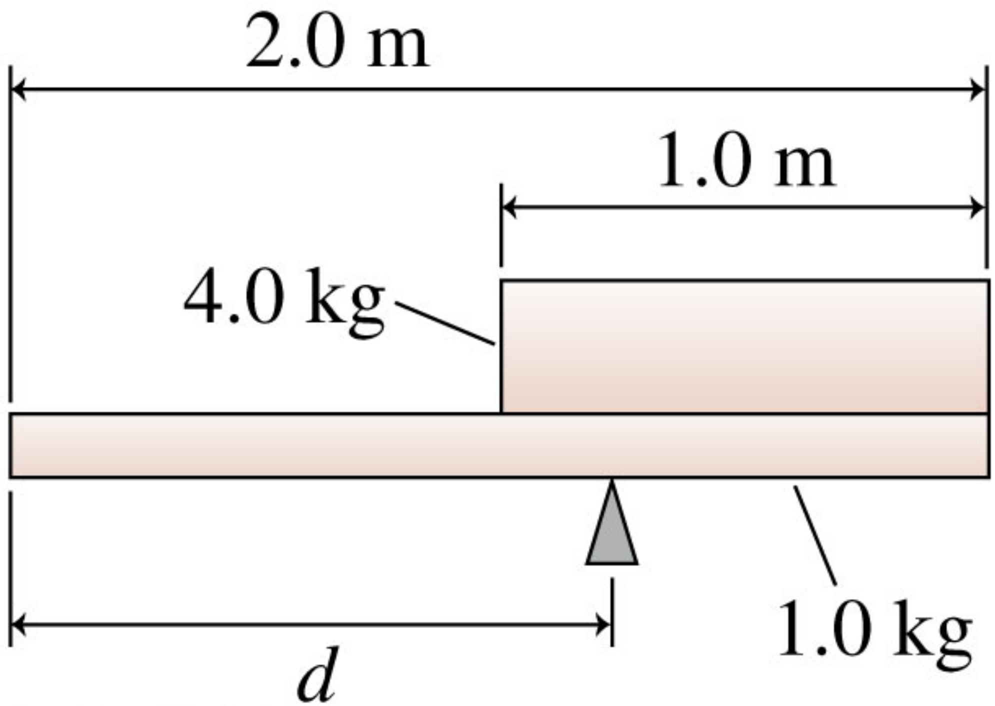
- Write equations for $\sum F_x = 0$, $\sum F_y = 0$, and $\sum \tau = 0$.
- Solve the three simultaneous equations.

ASSESS Check that your result is reasonable and answers the question.

Whiteboard Problem 8: Problem 12-29

The two objects are balanced on the pivot. What is distance d?

(Assume that the objects have uniform density, so their center of mass is located at their geometric center)



Whiteboard Problem 8: Problem 12-29

The two objects balanced on the pivot. What is distance d?

(Assume that the objects have uniform density, so their center of mass is located at their geometric center)

$\vec{\tau} = \vec{r} \times \vec{F}$

$|\vec{\tau}| = rF\sin\theta$ 90°

$= rF$

$\Sigma \tau_o = 0$

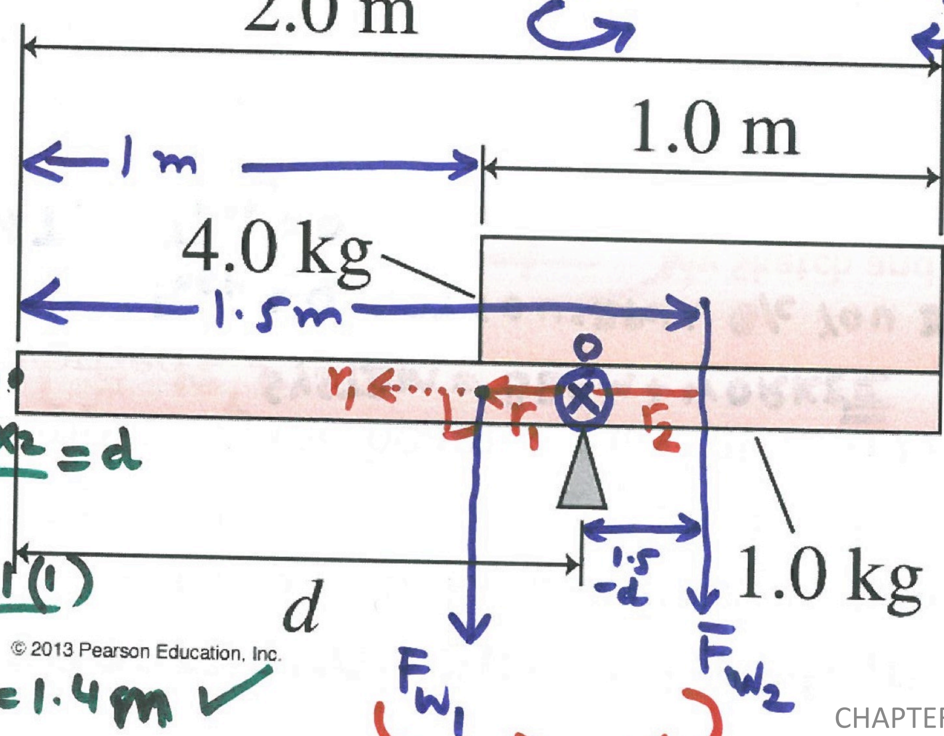
$\cancel{9.8}(1)(d-1) - \cancel{9.8}(4)(1.5-d) = 0$

$\cancel{d-1-1.5+d=0}$

$d-1-6+4d=0$

$5d=7$

$\Rightarrow d = 1.4\text{ m.}$ ✓



Find c.m. & demand that it lies @ fulcrum.

CHOOSE ORIGIN

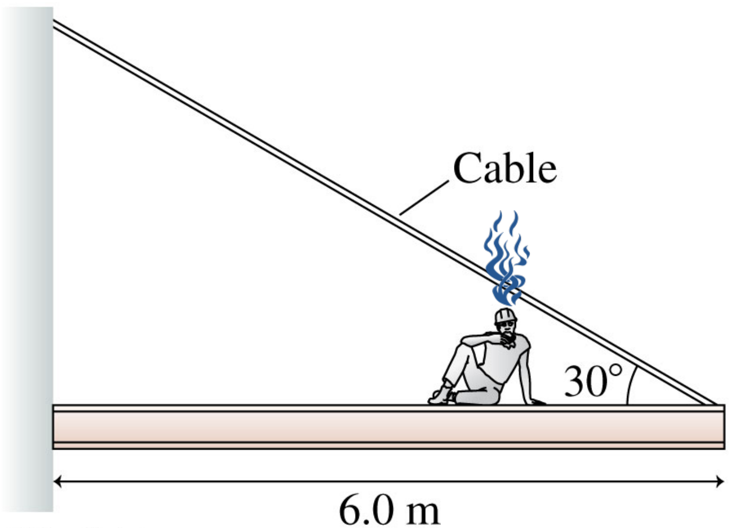
$x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = d$

$= \frac{4(1.5)+1(1)}{4+1}$

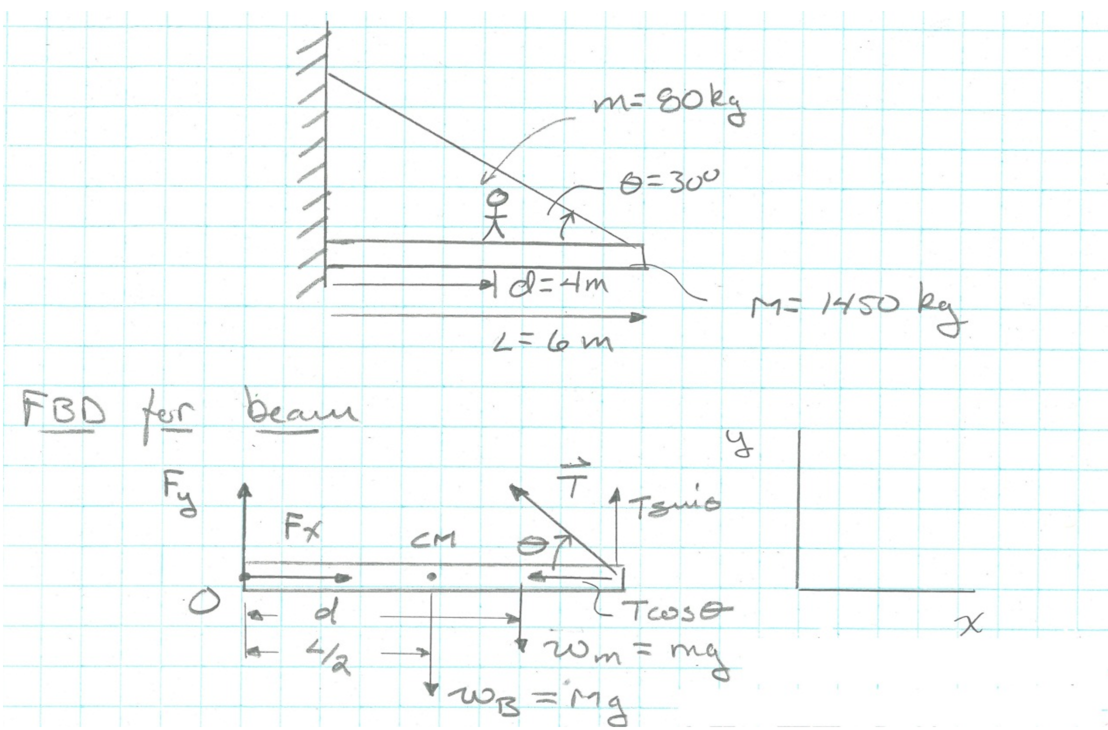
$= 7/5 = 1.4\text{ m}$ ✓

Whiteboard Problem 9: Problem 12-59

an 80 kg construction worker sits down 2.0 m from the end of a 1450 kg steel beam to eat his lunch. The cable supporting the beam is rated at 15,000 N. Should the worker be worried?



My Sketch and FBD:



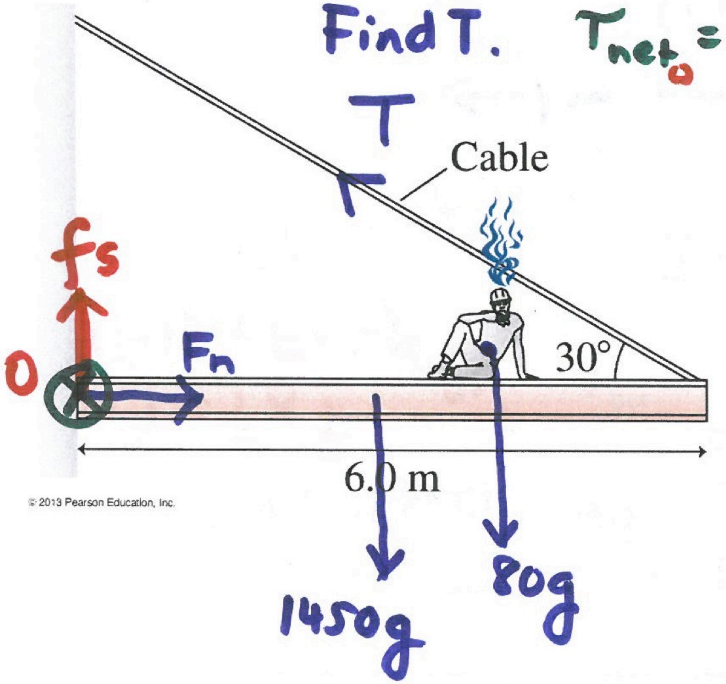
Whiteboard Problem 9: Problem 12-62

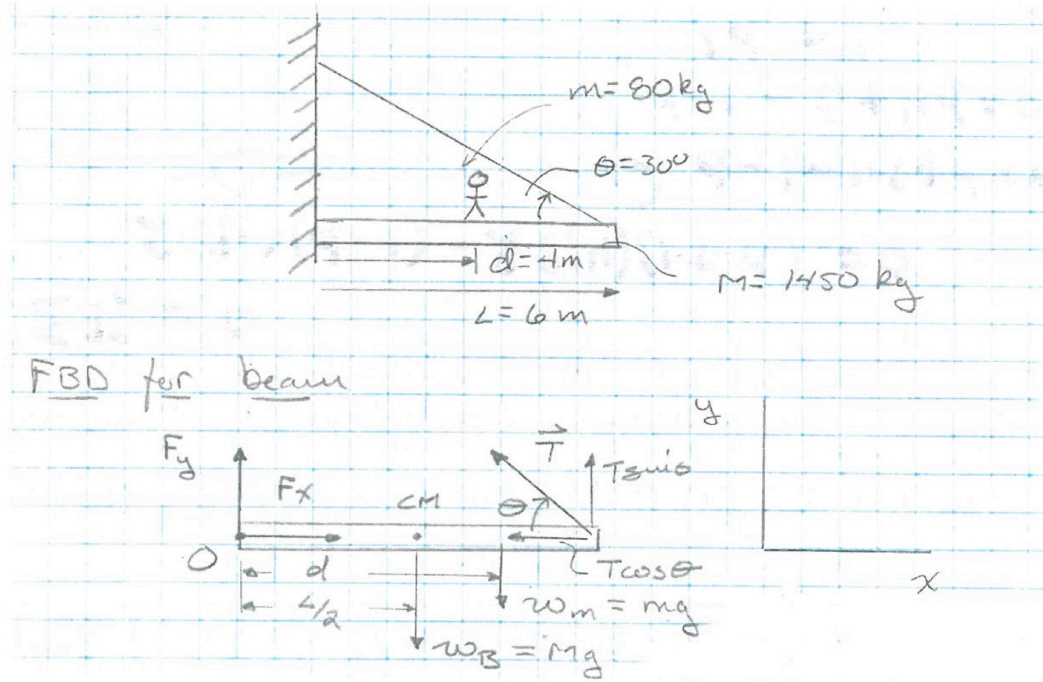
an 80 kg construction worker sits down 2.0 m from the end of a 1450 kg steel beam to eat his lunch. The cable supporting the beam is rated at 15,000 N. Should the worker be worried? SYSTEM = BEAM + WORKER

$F_{net} = 0$ NOT SO USEFUL B/c YOU DON'T KNOW f_s & F_n.

Find T. $T_{net_O} = 0$

My Sketch and FBD:

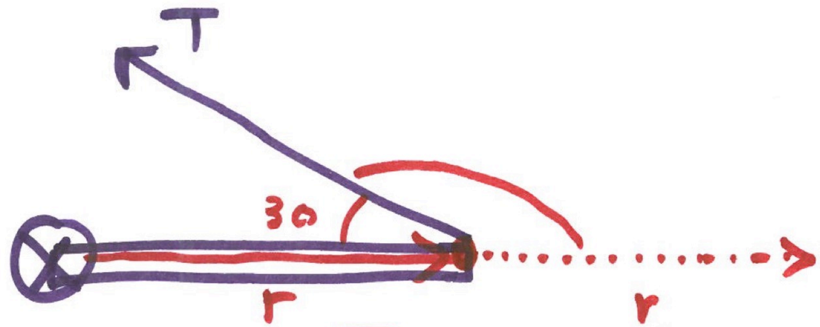




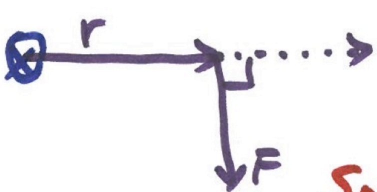
- CHOOSE PIVOT AXIS FOR $\boxed{\tau_{net} = 0}$ CALCULATION.
- SYSTEM = worker + beam
- Find T. If > 15000N

$F_{net} = 0$ X

$1450(9.8)(3)\cancel{\sin 90} + 80(9.8)(4) - \boxed{(T\sin 30)(6)} = 0$



$T\,r\,\sin 150^\circ$
$= T(6)\sin 30^\circ$

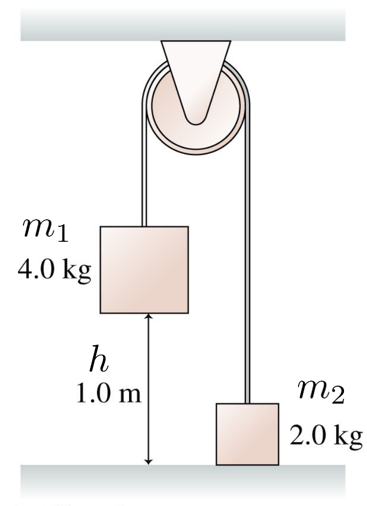


Solve for T:

T = 15,255 N

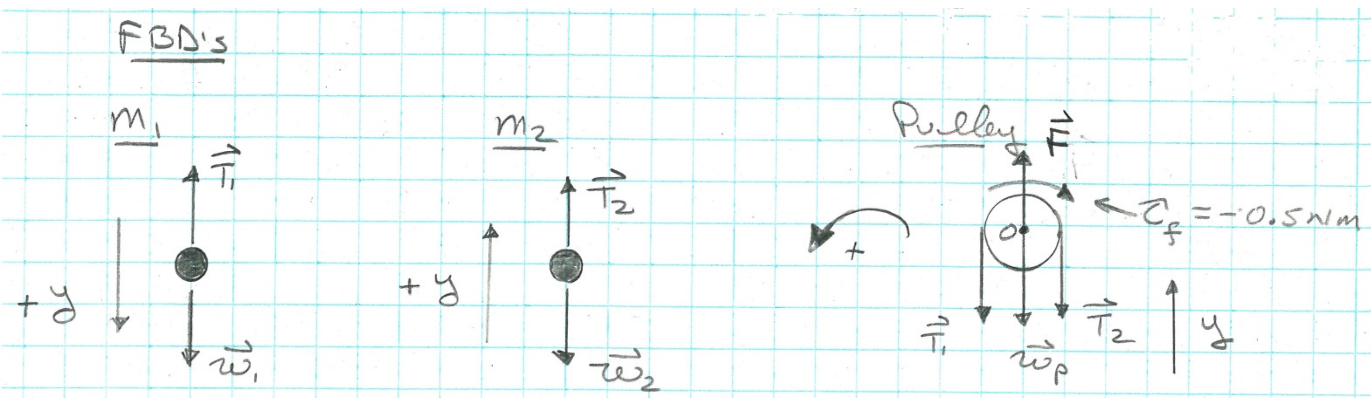
Whiteboard Problem 10: Problem 12-86

The figure shows two blocks connected over a pulley by a massless rope.
If the 4 kg block is released from rest, find the time for it to hit the floor if:



1. The **pulley is an ideal massless frictionless pulley**.
   [Hint: Find the acceleration, then use kinematics]

2. The **pulley is a real disk** 12 cm in diameter with a mass of 2 kg. Also, as a real pulley, as it turns, there is a friction torque of 0.5 Nm on the axle.
   (Hint: for a real pulley, the tension in the rope will be different on the two sides of the pulley – otherwise, it wouldn't rotate!)

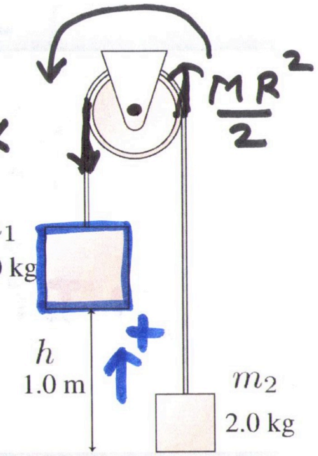


Whiteboard Problem 10: Problem 12-86

The figure shows two blocks connected over a pulley by a massless rope. If the 4 kg block is released from rest, find the time for it to hit the floor if:

1. The **pulley is an ideal massless frictionless pulley**. [Hint: Find the acceleration, then use kinematics] $a = 3.27\,m/s^2$ $t = 0.78\,s$
2. The **pulley is a real disk** 12 cm in diameter with a mass of 2 kg. Also, as a real pulley, as it turns, there is a friction torque of 0.5 Nm on the axle. $a = ?$ $1.6\,m/s^2$
   (Hint: for a real pulley, the tension in the rope will be different on the two sides of the pulley – otherwise, it wouldn't rotate!)

α, a $\quad a_t = R\alpha$ $\quad \frac{MR^2}{2}$

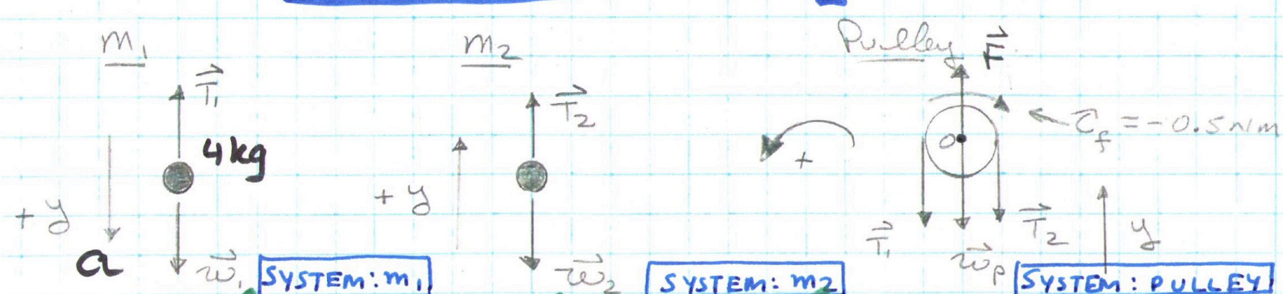


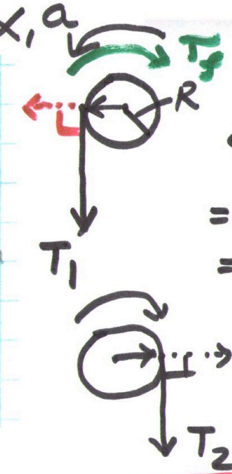
$V_i = 0$

$\Delta y = \cancel{v_i \Delta t} + \frac{1}{2} a (\Delta t)^2$

$-1 = \frac{1}{2}(-1.6)(\Delta t)^2$

$\boxed{\Delta t = 1.1\,sec}$





$r \times T_1$

$= (R)(T_1)\sin 90$

$= RT_1$

RT_2

$4g - T_1 = 4a$

$\Sigma F = ma$

$T_1 = 4g - 4a$

$T_2 - 2g = 2a$

$T_2 = 2g + 2a$

$\Sigma_0 \tau = I\alpha$

$$\boxed{\underbrace{RT_1 - RT_2}_{R(T_1 - T_2)} - T_f = \left(\frac{MR^2}{2}\right)\left(\frac{a}{R}\right)}$$

$(0.06)[4g - 4a - 2g - 2a] - 0.5 = \frac{1}{2}(2)(0.06)\,a$

$\Rightarrow 0.06(2g) - 6(0.06)a - 0.5 = 0.06a$

$\Rightarrow a = 0.676/0.42 = 1.6\,m/s^2$ Now that you know 'a', can find 'Δt' using Kinematics as shown up above.

SUPPOSE PULLEY HAS MASS BUT NO FRICTION!

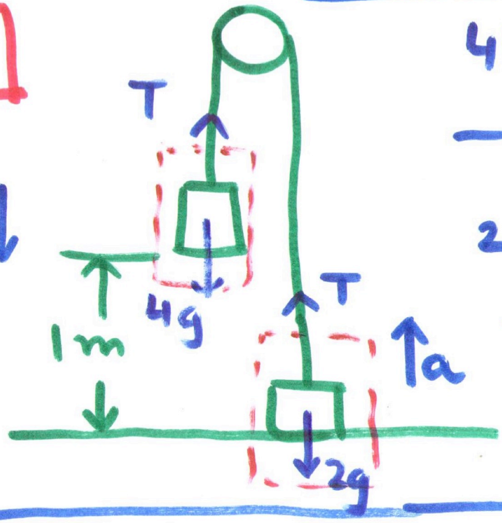
$$T_1 R - T_2 R = I\alpha$$

$$T_1 \cancel{R} - T_2 \cancel{R} = \frac{M\cancel{R}^{2}}{2} \cdot \frac{a}{\cancel{R}}$$

$$T_1 - T_2 = \frac{Ma}{2}$$

SAME QUESTION BUT SHOWING THE WORK I DID IN THE 8 AM CLASS

1.



4kg block:

$\rightarrow 4g - T = 4a$

2kg block:

$\rightarrow T - 2g = 2a$

$4g - (2g + 2a) = 4a$

$\Rightarrow 2g = 6a \Rightarrow a = \frac{2(9.8)}{6}$

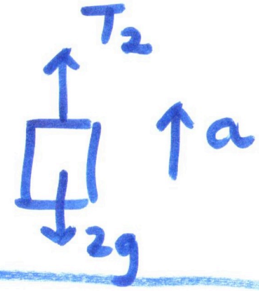
$T = 2g + 2a$

$= 3.27\ m/s^2$

2.

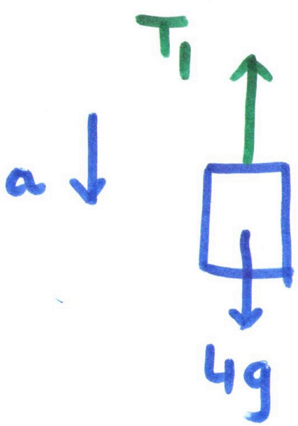
2kg block:

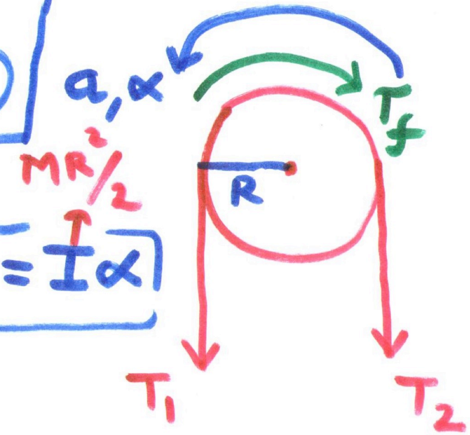
$\boxed{T_2 - 2g = 2a}$ — ②



4kg block:

$\boxed{4g - T_1 = 4a}$ — ①





$\Sigma T = I\alpha$

$\boxed{T_1R - T_2R - T_f = I\alpha}$

$T_1R - T_2R - 0.5 = \cancel{\left[\frac{2(0.06)^2}{2}\right]}\frac{a}{\cancel{R}}\ \frac{MR^{\cancel{2}}}{2}$

$\boxed{\underbrace{T_1R - T_2R}_{(T_1 - T_2)R} - 0.5 = \frac{MaR}{2}}$ — ③

A Comparison of Translation and Rotation Equations

Translation | **Rotation**

Kinematics

$$v = \frac{dx}{dt} \quad \& \quad a = \frac{dv}{dt}$$

$$\omega = \frac{d\theta}{dt} \quad \& \quad \alpha = \frac{d\omega}{dt}$$

Constant Acceleration

$$x_f = x_i + v_i \Delta t + \tfrac{1}{2} a (\Delta t)^2$$

$$\theta_f = \theta_i + \omega_i \Delta t + \tfrac{1}{2} \alpha (\Delta t)^2$$

$$v_f = v_i + a \Delta t$$

$$\omega_f = \omega_i + \alpha \Delta t$$

$$v_f^2 = v_i^2 + 2a(\Delta x)^2$$

$$\omega_f^2 = \omega_i^2 + 2\alpha(\Delta\theta)^2$$

Newton's 2nd Law

$$\vec{F}_{net} = m\vec{a}$$

$$\vec{\tau}_{net} = I\vec{\alpha}$$

Kinetic Energy

$$K = \tfrac{1}{2} m v^2$$

??

Momentum

$$\vec{p} = m\vec{v}$$

??

$$\vec{p}_{\text{final}} = \vec{p}_{\text{initial}}$$

??

A Comparison of Translation and Rotation Equations

	Translation	Rotation

Kinematics

Translation: $v = \frac{dx}{dt}$ & $a = \frac{dv}{dt}$

Rotation: $\omega = \frac{d\theta}{dt}$ & $\alpha = \frac{d\omega}{dt}$

Constant Acceleration

Translation:

$$x_f = x_i + v_i\Delta t + \frac{1}{2}a(\Delta t)^2$$
$$v_f = v_i + a\Delta t$$
$$v_f^2 = v_i^2 + 2a(\Delta x)^2$$

Rotation:

$$\theta_f = \theta_i + \omega_i\Delta t + \frac{1}{2}\alpha(\Delta t)^2$$
$$\omega_f = \omega_i + \alpha\Delta t$$
$$\omega_f^2 = \omega_i^2 + 2\alpha(\Delta\theta)^2$$

Newton's 2nd Law

Translation: $\vec{F}_{net} = m\vec{a}$

Rotation: $\vec{\tau}_{net} = I\vec{\alpha}$

Kinetic Energy

Translation: $K = \frac{1}{2}mv^2$

Rotation: ?? $K_{rot} = \frac{1}{2}I\omega^2$

Momentum

Translation: $\vec{p} = m\vec{v}$

Rotation: ?? $\vec{L} = I\vec{\omega}$

Translation: $\vec{p}_{\text{final}} = \vec{p}_{\text{initial}}$

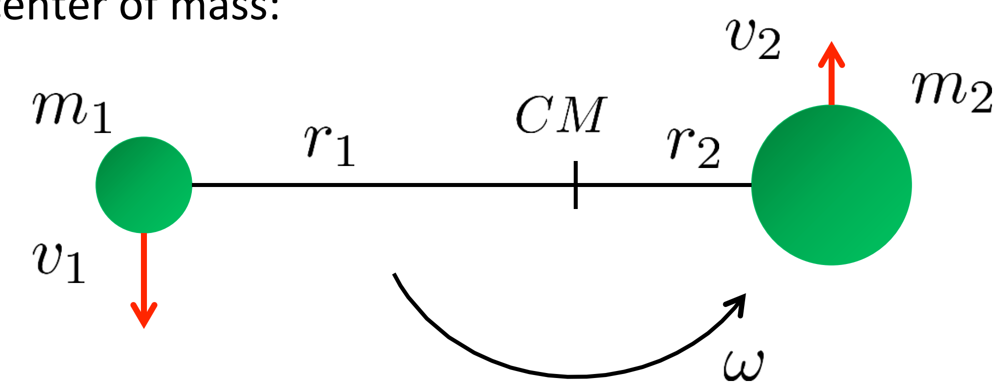
if no external forces act [System is isolated]

Rotation: C.O.A.M. ?? $\vec{L}_i = \vec{L}_f$

if no external torques act [System is isolated]

Rotational Kinetic Energy (Sec. 12.3)

Consider two mass points connected by a massless rigid rod rotating about the center of mass:



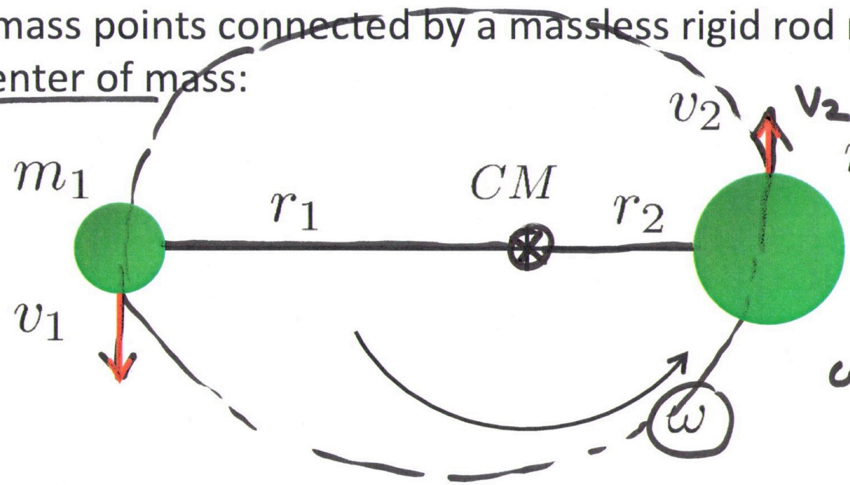
The total kinetic energy is:

$$K = \tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2$$
$$= \tfrac{1}{2}m_1r_1^2\omega^2 + \tfrac{1}{2}m_2r_2^2\omega^2$$
$$= \tfrac{1}{2}\underbrace{(m_1r_1^2 + m_2r_2^2)}_{I,\ \text{the moment of inertia}}\omega^2$$

So, the Rotational Kinetic Energy is: $K_{\text{rot}} = \tfrac{1}{2}I\omega^2$

Rotational Kinetic Energy (Sec. 12.3)

Consider two mass points connected by a massless rigid rod rotating about the center of mass:



ω is same!

$V = r\omega$

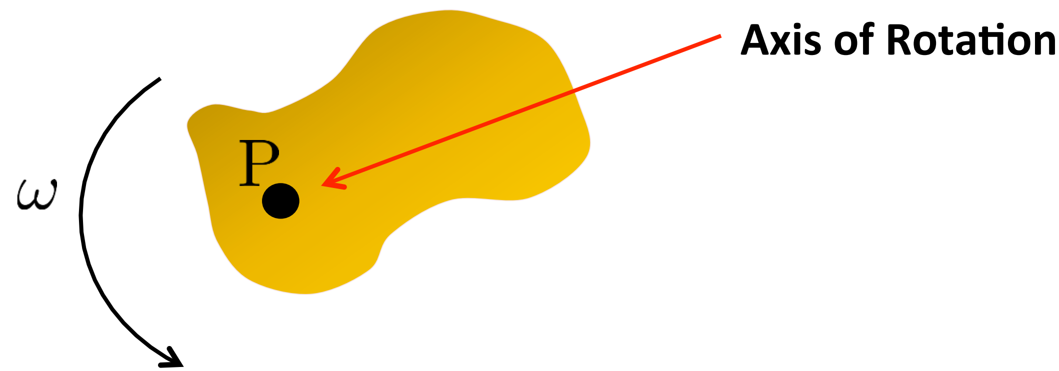
The total kinetic energy is:

$$
\begin{aligned}
K &= \tfrac{1}{2}m_1 \underline{\underline{v_1}}^2 + \tfrac{1}{2}m_2 \underline{v_2}^2 \\
&= \tfrac{1}{2}m_1 \underline{\underline{r_1^2\omega^2}} + \tfrac{1}{2}m_2 \underline{\underline{r_2^2\omega^2}} \\
&= \tfrac{1}{2}\underbrace{(m_1 r_1^2 + m_2 r_2^2)}_{I,\ \text{the moment of inertia}}\omega^2 = \tfrac{1}{2} I\omega^2
\end{aligned}
$$

So, the Rotational Kinetic Energy is: $K_{\text{rot}} = \frac{1}{2} I\omega^2$

Rotational Kinetic Energy (Sections 12.3 and 12.9)

We can generalize this to any rotating rigid body:



$$\text{Rotational Kinetic Energy, } K_{\text{rot}} = \frac{1}{2} I_p \omega^2$$

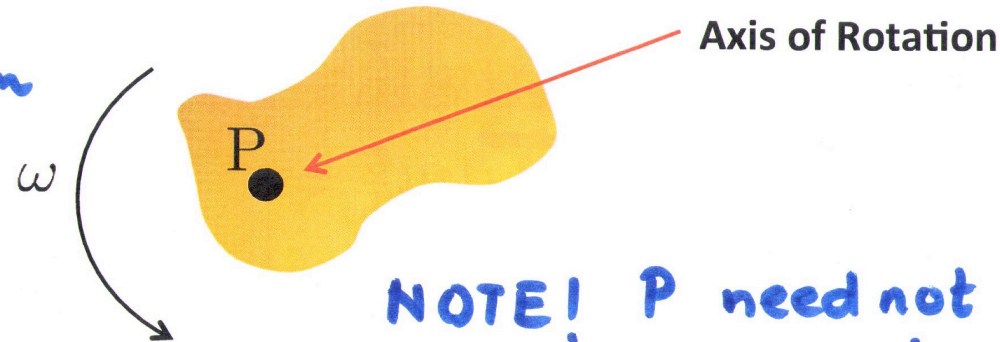
Where: I_p = Moment of inertia about the rotation axis, P

Compare this to $\frac{1}{2}mv^2$ for translational motion. Also note that a rigid body can have two forms of kinetic energy: translational, $\frac{1}{2}mv^2$, and rotational, $\frac{1}{2}I_p\omega^2$.

Rotational Kinetic Energy (Sections 12.3 and 12.9)

We can generalize this to any rotating rigid body:

If you know I_{cm} & need to find I about P, then use //-Axis THEOREM.



NOTE! P need not be c.m. !

Rotational Kinetic Energy, $K_{\text{rot}} = \frac{1}{2} I_p \omega^2$

Where: I_p = Moment of inertia about the rotation axis, P

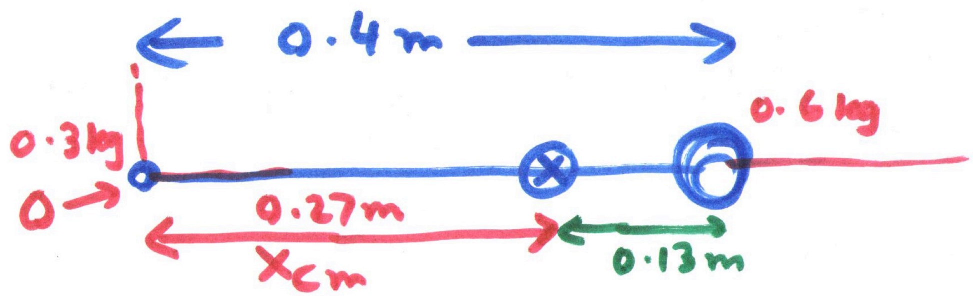
Compare this to $\frac{1}{2}mv^2$ for translational motion. Also note that a rigid body can have two forms of kinetic energy: translational, $\frac{1}{2}mv^2$, and rotational, $\frac{1}{2}I_p\omega^2$.

Whiteboard Problem 11: Problem 12-49

A 300 g ball and a 600 g ball are connected by a 40-cm-long massless, rigid rod. The structure rotates about its center of mass at 100 rpm. What is its rotational kinetic energy?

Whiteboard Problem 11: Problem 12-49

A 300 g ball and a 600 g ball are connected by a 40-cm-long massless, rigid rod. The structure rotates about its center of mass at 100 rpm. What is its rotational kinetic energy?



$$x_{cm} = \frac{0.3(0) + 0.6(0.4)}{0.3 + 0.6} = 0.27\text{m}$$

$$KE_{rot} = \frac{1}{2} I \omega^2 \qquad \omega = \frac{100 \cdot 2\pi}{60}\ \text{rad s}^{-1} = 10.47\ \text{rad s}^{-1}$$

$$I = \sum_{i=1}^{2} m_i r_i^2 = m_1 r_1^2 + m_2 r_2^2 = 0.3(0.27^2) + 0.6(0.13^2)$$

$$= 0.032\ \text{kgm}^2$$

$$\frac{1}{2}(0.032)(10.47) = 1.75\ \text{J}$$

Rolling Motion and Friction (Sec. 12.9)

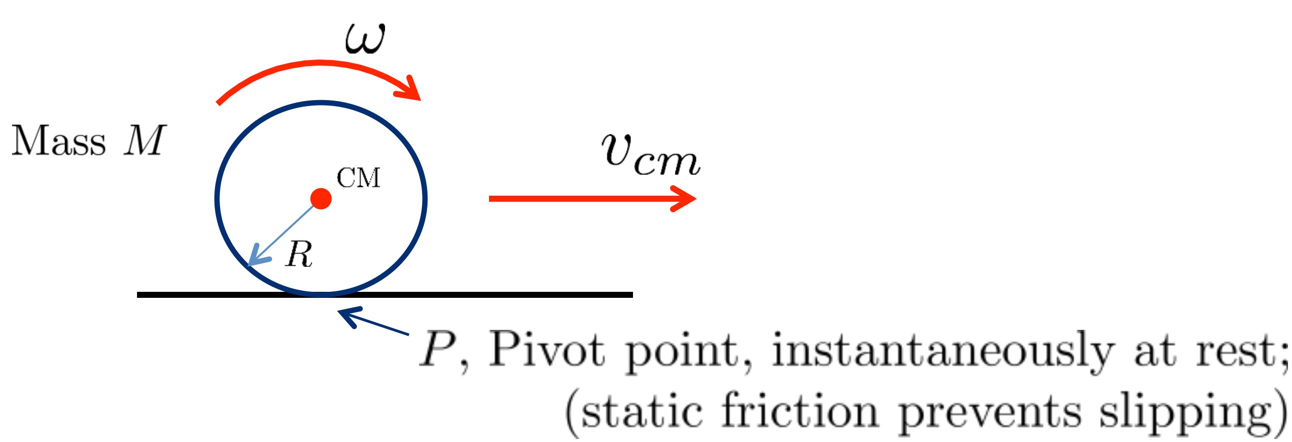
Take a nice round sphere and set it rolling.
Q: Can you even get it to start rolling if the ground were perfectly frictionless?
Q: If it can roll, it may keep rolling practically forever! Isn't any energy dissipated?

Conclusion:

Friction is ________________ ____ ____________ ______________ !!

In the case of rolling without sliding, friction _____________ ____ ________________

___ __________ !!!



Rolling Motion and Friction (Sec. 12.9)

Take a nice round sphere and set it rolling.

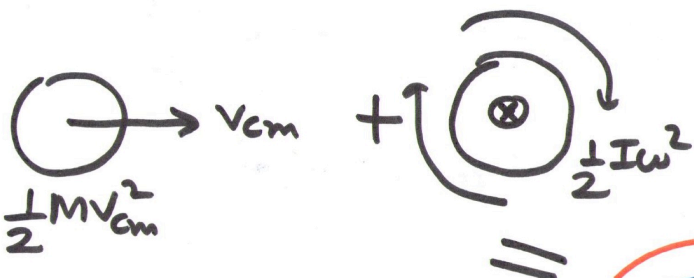
Q: Can you even get it to start rolling if the ground were perfectly frictionless?

Q: If it can roll, it may keep rolling practically forever! Isn't any energy dissipated?

Conclusion:

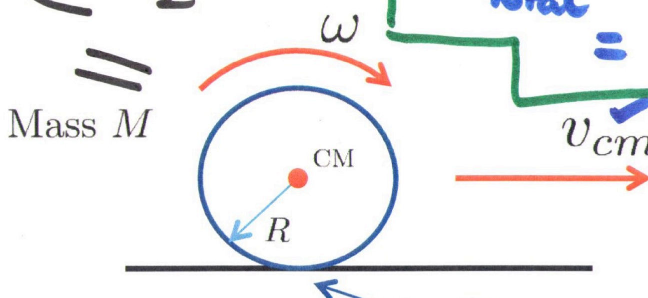
Friction is necessary to cause rolling !!

In the case of rolling without sliding, friction causes no dissipation of energy !!!



$$\boxed{\begin{aligned} KE_{total} &= KE_{rot} + KE_{transl} \\ &= \tfrac{1}{2}I\omega^2 + \tfrac{1}{2}Mv_{cm}^2 \end{aligned}}$$

where, for rolling without sliding, $V_{cm} = R\omega$

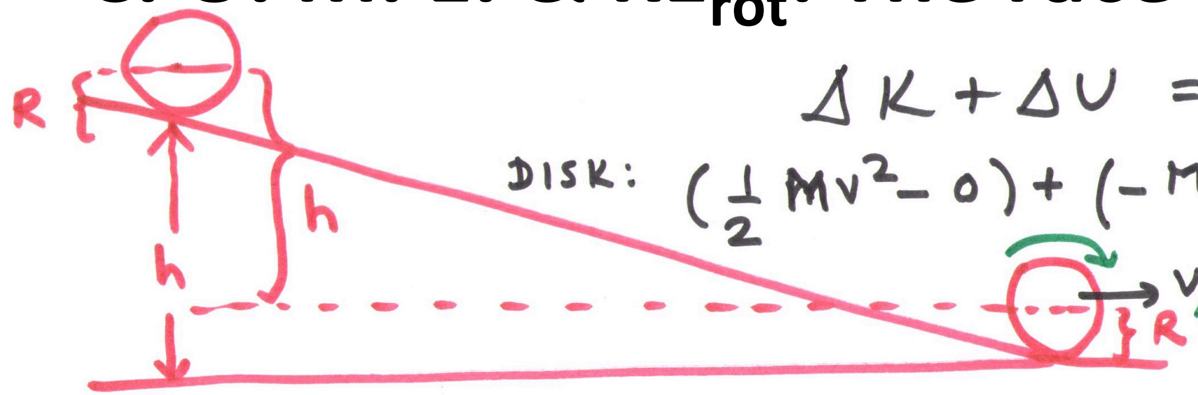


P, Pivot point, instantaneously at rest;
(static friction prevents slipping)

C. O. M. E. & KE_{rot}: The race downhill

$\Delta K + \Delta U = 0$ (b/c $W_{nc} = 0$)

DISK: $\left(\frac{1}{2}Mv^2 - 0\right) + (-Mgh)$ X



$V = r\omega$

DISK: $\Delta K_{rot} + \Delta K_{trans} + \Delta U = 0$

$$\left(\frac{1}{2}I\omega^2 - 0\right) + \left(\frac{1}{2}Mv^2 - 0\right) + (-Mgh) = 0$$

$$\frac{1}{2}\left(\frac{\cancel{M}\cancel{R^2}}{2}\right)\left(\frac{v}{\cancel{R}}\right)^2 + \frac{1}{2}\cancel{M}v^2 = \cancel{M}gh$$

[Mass, Radius, CANCEL!]

$$\frac{1}{4}v^2 + \frac{1}{2}v^2 = gh \Rightarrow \frac{3}{4}v^2 = gh \Rightarrow v = \sqrt{\frac{4}{3}gh}$$

RING: $\frac{1}{2}I\omega^2 + \frac{1}{2}Mv^2 + (-Mgh) = 0$

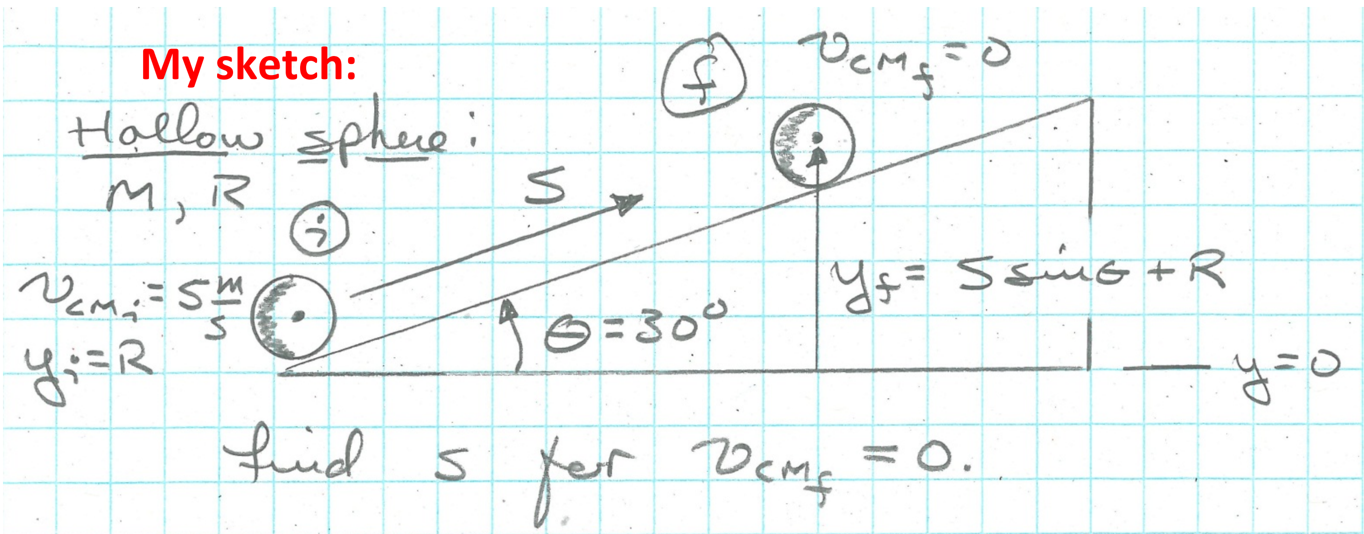
$$\frac{1}{2}(\cancel{M}\cancel{R^2})\left(\frac{v}{\cancel{R}}\right)^2 + \frac{1}{2}\cancel{M}v^2 = \cancel{M}gh$$

$$\frac{1}{2}v^2 + \frac{1}{2}v^2 = gh \Rightarrow v^2 = gh \Rightarrow v = \sqrt{gh}$$

Whiteboard Problem 12: Problem 12-69

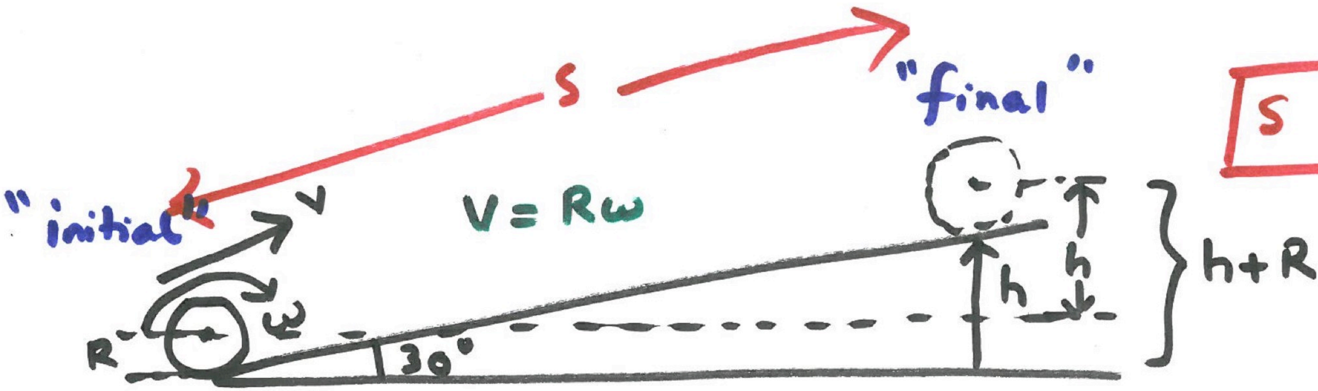
A hollow sphere is rolling along a horizontal floor at 5.0 m/s when it comes to a 30° incline. How far up the incline does it roll before reversing direction?

My sketch:



Repeat the calculation for a sliding block sliding up the same frictionless incline with the same initial speed. Why do you get different answers, and do the numbers make sense?

Whiteboard Problem 11: Problem 12-69



$$\boxed{s \sin 30 = h}$$

HOLLOW SPHERE : $I = \frac{2}{3}MR^2$

$$\Delta K_{tr} + \Delta K_{rot} + \Delta U = \cancel{W_{nc}}$$

$$\left(0 - \frac{1}{2}Mv^2\right) + \left(0 - \frac{1}{2}I\omega^2\right) + mgh = 0$$

$$-\frac{1}{2}\cancel{M}v^2 - \frac{1}{\cancel{2}}\cdot\left(\frac{\cancel{2}}{3}\cancel{M}\cancel{R^2}\right)\left(\frac{v^2}{\cancel{R^2}}\right) + \cancel{m}gh = 0$$

$$\left(-\frac{1}{2} - \frac{1}{3}\right)v^2 + gh = 0 \Rightarrow gh = \frac{5}{6}v^2$$

$$h = \frac{5v^2}{6g} = \frac{25}{12} = 2.1 \text{ m}$$

$$\therefore s = \frac{h}{\sin 30} = 4.2 \text{ m}$$

A Comparison of Translation and Rotation Equations

Translation | **Rotation**

Kinematics

$$v = \frac{dx}{dt} \quad \& \quad a = \frac{dv}{dt}$$

$$\omega = \frac{d\theta}{dt} \quad \& \quad \alpha = \frac{d\omega}{dt}$$

Constant Acceleration

$$x_f = x_i + v_i \Delta t + \tfrac{1}{2} a (\Delta t)^2$$

$$\theta_f = \theta_i + \omega_i \Delta t + \tfrac{1}{2} \alpha (\Delta t)^2$$

$$v_f = v_i + a \Delta t$$

$$\omega_f = \omega_i + \alpha \Delta t$$

$$v_f^2 = v_i^2 + 2a(\Delta x)^2$$

$$\omega_f^2 = \omega_i^2 + 2\alpha(\Delta\theta)^2$$

Newton's 2nd Law

$$\vec{F}_{net} = m\vec{a}$$

$$\vec{\tau}_{net} = I\vec{\alpha}$$

Kinetic Energy

$$K = \tfrac{1}{2} m v^2$$

$$K_{\rm rot} = \tfrac{1}{2} I \omega^2$$

Momentum

$$\vec{p} = m\vec{v}$$

??

$$\vec{p}_{\rm final} = \vec{p}_{\rm initial}$$

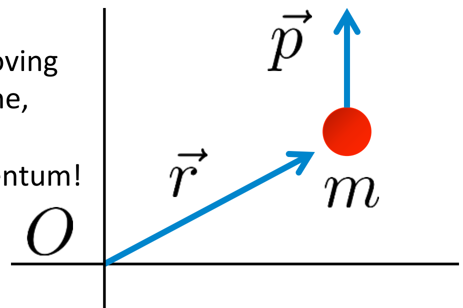
??

Angular Momentum (Sec. 12.11)

When you have rotational motion, we've seen that you have rotational kinetic energy. We also have the rotational analog of momentum, **angular momentum**.

For a particle:

Note!
You can be moving in a straight line, yet have an angular momentum!



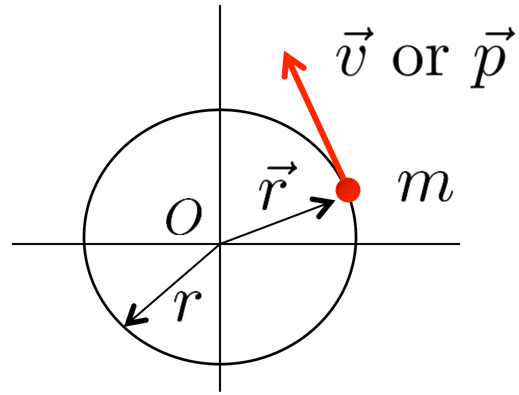
Angular Momentum about Point O:

$$\vec{L} = \vec{r} \times \vec{p}$$

Magnitude:

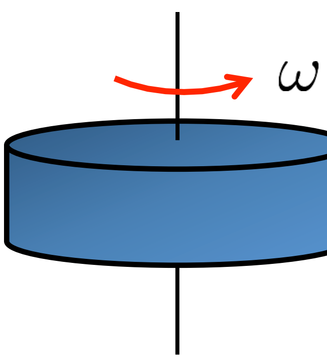
Direction:

Uniform Circular Motion:



$$|\vec{L}| = |\vec{r} \times \vec{p}| = rp = mrv = mr^2\omega$$

Rigid Body rotating about a fixed symmetry axis:



$$\vec{L} = I\vec{\omega}$$

Angular Momentum Conservation

Starting with $\vec{L} = \vec{r} \times \vec{p}$:

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p}) = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}$$

$$= \cancel{\vec{v} \times m\vec{v}} + \vec{r} \times \vec{F}_{net}$$

$$= \vec{\tau}_{net} \text{ (the net torque)}$$

So, we have another form of Newton's 2nd Law for Rotation: $\vec{\tau}_{net} = \frac{d\vec{L}}{dt}$

Now, if there are no external torques acting on a system, then:

$$\vec{\tau}_{net} = 0 = \frac{d\vec{L}}{dt} \Rightarrow \vec{L} = \text{Constant}$$

In all of the problems that we'll do, we'll apply conservation of momentum like this:

$$\vec{L}_{\text{final}} = \vec{L}_{\text{initial}}$$

And remember, the vector sign just indicates positive for CCW and negative for CW.

We can now finish the table that we started at the beginning

A Comparison of Translation and Rotation Equations

Translation | **Rotation**

Kinematics

$$v = \frac{dx}{dt} \quad \& \quad a = \frac{dv}{dt}$$

$$\omega = \frac{d\theta}{dt} \quad \& \quad \alpha = \frac{d\omega}{dt}$$

Constant Acceleration

$$x_f = x_i + v_i \Delta t + \frac{1}{2} a (\Delta t)^2$$

$$v_f = v_i + a \Delta t$$

$$v_f^2 = v_i^2 + 2a(\Delta x)^2$$

$$\theta_f = \theta_i + \omega_i \Delta t + \frac{1}{2} \alpha (\Delta t)^2$$

$$\omega_f = \omega_i + \alpha \Delta t$$

$$\omega_f^2 = \omega_i^2 + 2\alpha(\Delta\theta)^2$$

Newton's 2nd Law

$$\vec{F}_{net} = m\vec{a}$$

$$\vec{\tau}_{net} = I\vec{\alpha}$$

Kinetic Energy

$$K = \frac{1}{2} m v^2$$

$$K_{\text{rot}} = \frac{1}{2} I \omega^2$$

Momentum

$$\vec{p} = m\vec{v}$$

$$\vec{p}_{\text{final}} = \vec{p}_{\text{initial}}$$

$\vec{L} = \vec{r} \times \vec{p}$ (particle)

$\vec{L} = I\vec{\omega}$ (rigid body)

$$\vec{L}_{\text{final}} = \vec{L}_{\text{initial}}$$

Conservation of Angular Momentum (Sec. 12.11)
Example: Skaters and Spinning

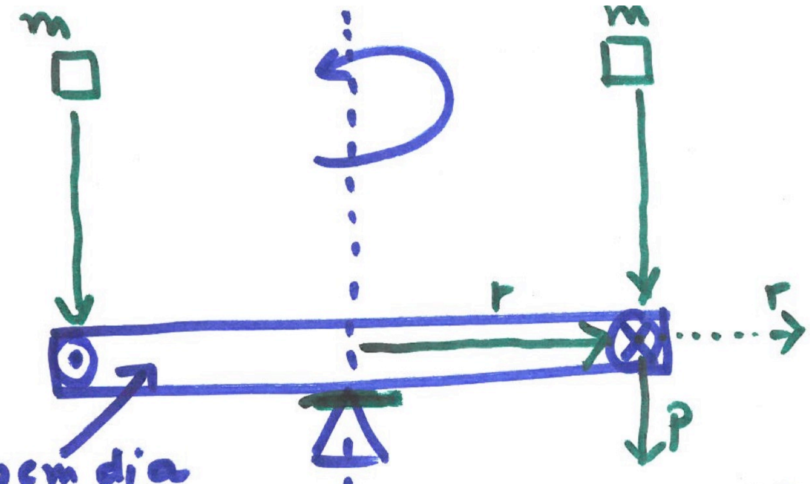
I'm sure that you all have seen spinning figure skaters. Here's Dorothy Hamill sometime in the 1980's, and here's Natalia Kanounnikov setting the spin world record.

How do the skaters spin so fast? They are decreasing their moment of inertia, and, *in the absence of any external torque,* conservation of angular momentum says the spin rate must go up!

Whiteboard Problem 13: Problem 12-46

A 2.0 kg, 20-cm-diameter turntable rotates at 100 rpm on frictionless bearings. Two 500 g blocks fall from above, hit the turntable simultaneously at opposite ends of a diameter, and stick. What is the turntable's angular velocity, in rpm, just after this event?

Whiteboard Problem 13: Problem 12-46



$\omega_f = ?$

$L_{i,\text{turntable}} \uparrow$

m : $\uparrow$ or $\downarrow$?

↳ $L_{i\,\text{masses}}$ ⊗

$\omega_i = 100\,\text{rpm}$

$= \frac{100 \cdot 2\pi}{60}\ \text{rads}^{-1}$

$I_{disk} = \frac{MR^2}{2} = 2\,\frac{(0.1\,m)^2}{2}$

$= 0.01\,\text{kgm}^2$

$L_i = I_{disk}\,\omega_i = (0.01)\left(\frac{100 \cdot 2\pi}{60}\right)$

$I_{disk} = \frac{MR^2}{2} = 0.01\,\text{kgm}^2$

$I_{masses} = mr^2 \times 2$

$= (0.5)(0.1)^2 \times 2$

$= 0.01$

$L_i = L_f$

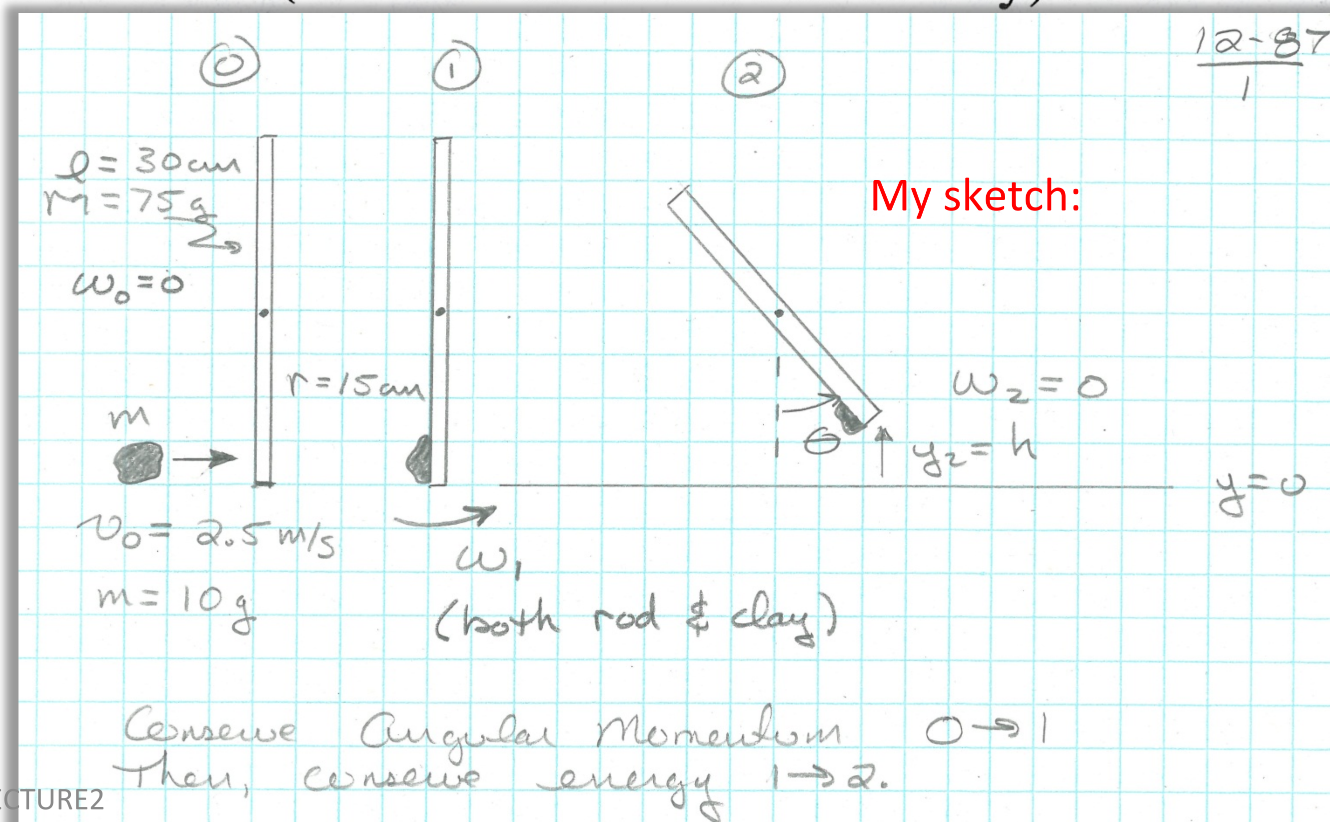
$(0.01)\left(\frac{100 \cdot \cancel{2\pi}}{\cancel{60}}\right) = I_{disk}\,\omega_f + I_{masses}\,\omega_f$

$= (0.01 + 0.01)\left(\omega_f \cdot \frac{\cancel{2\pi}}{\cancel{60}}\right)$

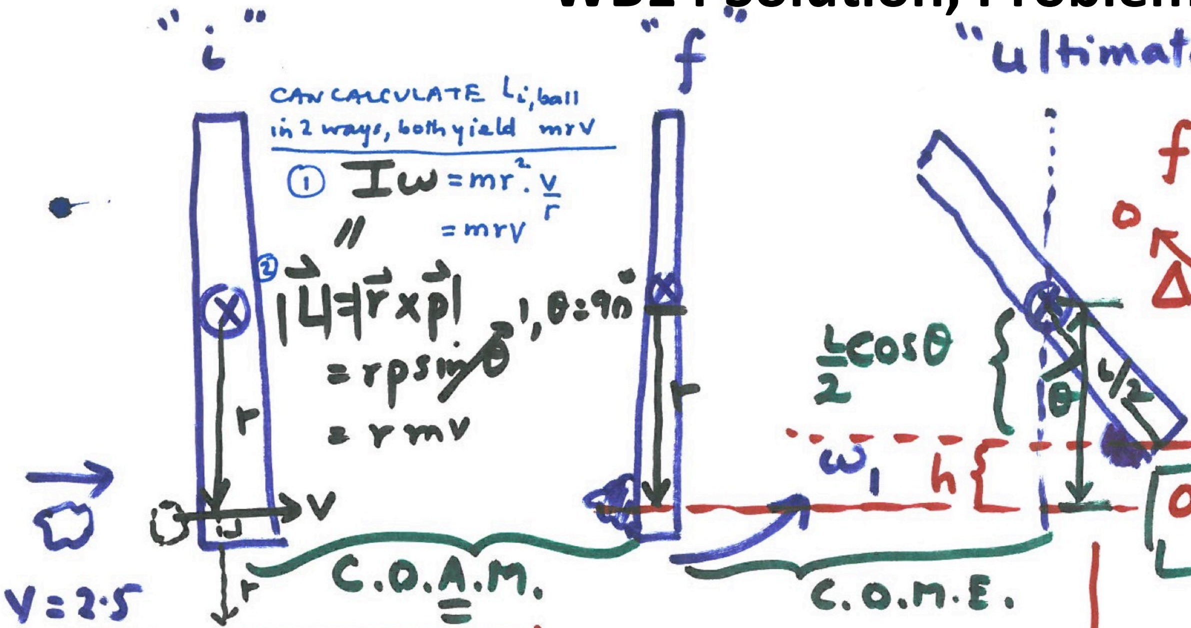
$\omega_f = 5.25\,\text{rads}^{-1}$; $\omega_f = 50\,\text{rpm}$

Whiteboard Problem 14: Problem 12-90

A 75 g, 30-cm-long rod hangs vertically on a frictionless, horizontal axle passing through its center. A 10 g ball of clay traveling horizontally at 2.5 m/s hits and sticks to the very bottom tip of the rod. To what maximum angle, measured from vertical, does the rod (with the attached ball of clay) rotate?



WB14 Solution, Problem 12.90



CAN CALCULATE $L_{i,ball}$ in 2 ways, both yield mrv

① $I\omega = mr^2 \cdot \frac{v}{r} = mrv$

② $|\vec{L}| = |\vec{r} \times \vec{p}| = rp\cancel{\sin\theta}^{1,\ \theta = 90°} = rmv$

$v = 2.5$ m/s

C.O.A.M.

$i-f$: $\boxed{L_i = L_f}$

$$\underbrace{(0.15)(0.01)(2.5)}_{L_{i,ball}} + \cancel{L_{i,stick}}^{0} = \overbrace{\underbrace{mr^2}_{I_{ball}} \cdot \omega_1}^{L_{f,ball}} + \overbrace{\underbrace{\left(\frac{ML^2}{12}\right)}_{I_{stick,cm}} \cdot \omega_1}^{L_{f,stick}}$$

$$\therefore 0.15(0.01)(2.5) = \underbrace{\left[0.01(0.15^2) + \frac{(0.075)(0.3^2)}{12}\right]}_{I_{ball+stick}} \omega_1$$

$\omega_1 = 4.76 \text{ rads}^{-1}$

$I_{ball+stick} = 0.0007875 \text{ kgm}^2$

C.O.M.E.

C.O.M.E.

$f-u$: $\underbrace{\Delta K} + \Delta U = \cancel{W_{nc}}^{0}$

$\Delta K = \cancel{\Delta K_{tr}}^{0} + \underbrace{\Delta K_{rot}} = K_f - K_i$

$= 0 - \frac{1}{2} I \omega_1^2$

$= \boxed{0 - \frac{1}{2}\underbrace{[0.0007875]}_{I_{ball+stick}} 4.76^2}$ $= \Delta K$!

$\Delta U = \cancel{\Delta U_{stick}}^{0} + \Delta U_{ball}$

$= m_{ball}\, g\left[\frac{L}{2} - \frac{L}{2}\cos\theta\right]$

$\Delta U = \boxed{0.01(9.8)\left(\frac{0.3}{2}\right)(1 - \cos\theta)}$

USE $\Delta K + \Delta U = 0$ to obtain

$1 - \cos\theta = 0.6074$

$\theta = \cos^{-1}(0.3926)$

$= 66.88°$

or $\boxed{67°}$