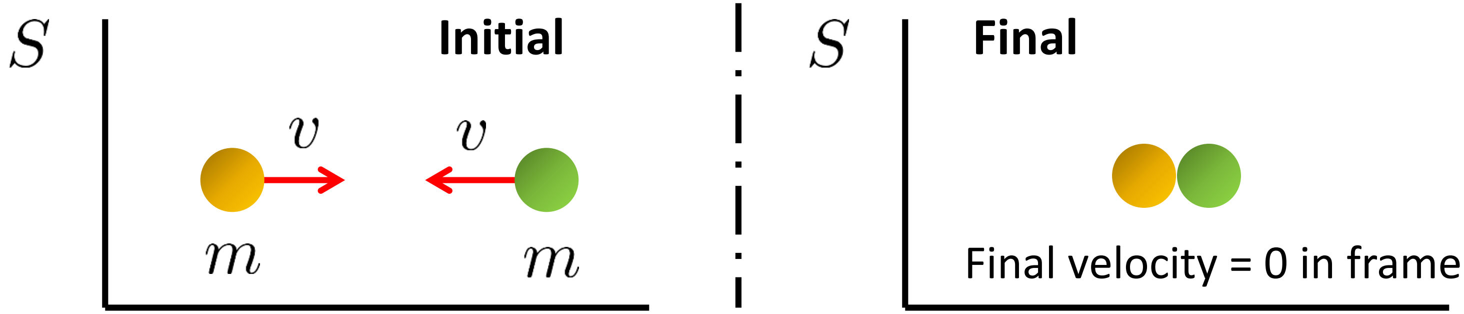
36-3: Dynamics in Special Relativity

So far in our study of SR we have concentrated on space, time, and velocity, i.e. kinematics. Einstein realized that SR also had to have a profound effect on dynamics as well.

But it is subtle to analyze: the central ingredient in Newtonian Mechanics is Newton's 2nd Law, F = ma, and accelerations are difficult to interpret using inertial reference frames.

A better approach is to use the concepts of Momentum and Energy, but even doing this leads to some surprises.

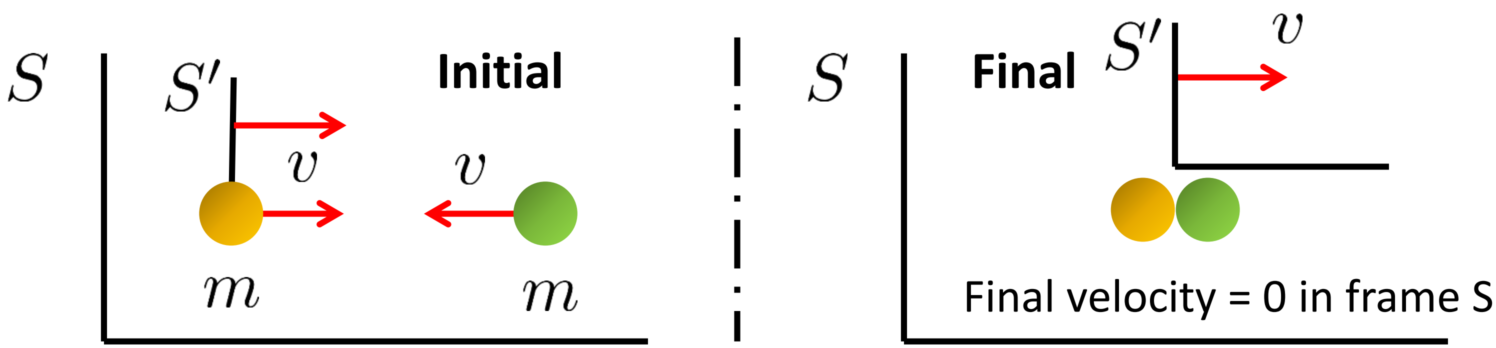
Consider a simple experiment: a 1D Perfectly Inelastic Collision of two identical masses:



Is momentum conserved in frame S? $p_i = p_f \Rightarrow mv - mv = 0$

So momentum is conserved in frame S

Dynamics in Special Relativity



Is momentum conserved in another inertial reference frame moving relative to S?

As an example: let's try frame S' that is moving with the orange mass before the collision and continues to move after the collision.

If: We use the GVT to find the velocities in S'

Then: Momentum is conserved in S'

But the GVT are not the correct transformations; they violate the 2nd postulate of SR! We should be using the Lorentz Velocity Transformations (LVT):

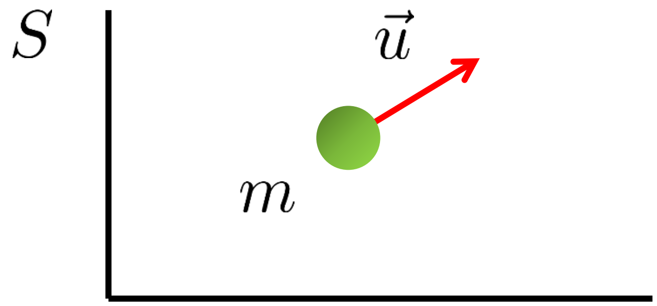
If: We use the LVT to find the velocities in S'

Then: Momentum is NOT conserved in S'

So, we have a problem here – is Momentum Conservation going to continue as a "Law of Physics," or should we abandon it?

The solution is that our definition of momentum is not correct.

Relativistic Momentum



The **Newtonian Momentum** of mass m is:

$$\vec{p} = m\vec{u}$$

But this is not conserved in all IRFs

Carefully studying simple two body collision problems using the LVT, like that above, reveals that **the quantity that is conserved in all IRFs is:**

The Relativistic Momentum

$$\vec{p} = \frac{m\vec{u}}{\sqrt{1 - u^2/c^2}} = \gamma_p m\vec{u}$$

Here the p subscript on the Lorentz factor means that it is calculated with the particle speed u – not the speed v between frame S and any other frame S'.

Also: For $u << c$, $\gamma_p \sim 1$, and $\vec{p} \to m\vec{u}$, the Newtonian Momentum

Whiteboard Problem 36-12

A 1.0 g particle has a momentum of 400,000 kg m/s.
What is the particle's speed? (LC)

Energy in SR *(wherein we encounter* ***E = mc²****)*

Consider a force acting on a mass in one dimension: **What is it's Kinetic Energy?**

Velocity, u = 0
Kinetic Energy, K = 0

Velocity, u
Kinetic Energy, K ?

Work Energy Theorem (from PHY181)

Change in Kinetic Energy = work done on m

$$\Delta K = K - 0 = \int \vec{F} \cdot d\vec{s} = \int F dx$$

Now, force can be written in terms of momentum (1D): $F = \frac{dp}{dt}$

So: $$K = \int \frac{dp}{dt} dx = \int \frac{dx}{dt} dp = \int u dp$$

Now in the Nonrelativistic Newtonian World: $p = mu \Rightarrow dp = mdu$

Therefore: $$K = \int_0^u umdu = m \int_0^u udu = \frac{1}{2} mu^2$$

Which is the Newtonian form of the KE that we used in PHY181.

Whiteboard Problem 36-13: Finding E = mc²

Continuing with what's on the previous slide, find an expression for the Kinetic Energy in the World of Einstein and Relativity. Starting point:

$$K = \int u\,dp \qquad p = \gamma_p m u = \frac{mu}{\sqrt{1 - u^2/c^2}}$$

Step 1: Find dp = some function of u times du: (LC)

$$\boxed{dp} = d\left(\frac{mu}{\sqrt{1-u^2/c^2}}\right)$$

Differentiate using the quotient rule

$$= \frac{\left[m(1-u^2/c^2)^{1/2}du - mu(\tfrac{1}{2})(1-u^2/c^2)^{-1/2}(-\tfrac{2u}{c^2})du\right]}{(1-u^2/c^2)}$$

Factor

$$= \frac{mdu}{(1-u^2/c^2)}\left\{(1-u^2/c^2)^{1/2} + \frac{u^2/c^2}{(1-u^2/c^2)^{1/2}}\right\}$$

Put over a common denominator

$$= \frac{mdu}{(1-u^2/c^2)}\left\{\frac{(1-u^2/c^2)^{1/2}(1-u^2/c^2)^{1/2} + u^2/c^2}{(1-u^2/c^2)^{1/2}}\right\}$$

Simplify

$$= \frac{mdu}{(1-u^2/c^2)}\left\{\frac{1-u^2/c^2+u^2/c^2}{(1-u^2/c^2)^{1/2}}\right\} = \boxed{\frac{mdu}{(1-u^2/c^2)^{3/2}}}$$

Whiteboard Problem 36-13: Finding E = mc^2

Step 2: Put your expression for dp into the integral for the Kinetic Energy

$$K = \int u\,dp = \int_0^u u \frac{m\,du}{(1 - u^2/c^2)^{3/2}}$$

Step 3: Integrate to find the Kinetic Energy K: (LC)

$$\boxed{K} = \int_0^u u \frac{m\,du}{(1 - u^2/c^2)^{3/2}} \quad = m \int_0^u u(1 - u^2/c^2)^{-3/2} du$$

$$= m \left(-\frac{c^2}{2}\right) \frac{(1 - u^2/c^2)^{-1/2}}{(-\frac{1}{2})} \Bigg|_0^u$$

$$= \frac{mc^2}{(1 - u^2/c^2)^{1/2}} \Bigg|_0^u$$

$$= \boxed{\frac{mc^2}{\sqrt{1 - u^2/c^2)}} - mc^2}$$

No need for integral tables! (Or you can use a table.)

Whiteboard Problem 36-13: Finding E = mc^2

Step 4: Interpret what we've got:

We have found that the Kinetic Energy of an object with mass m and speed u is:

$$K = \frac{mc^2}{\sqrt{1 - u^2/c^2}} - mc^2$$

Define: The Total Relativistic Energy:

$$E = \frac{mc^2}{\sqrt{1 - u^2/c^2}} = \gamma_p mc^2$$

*(Note: this **not** the same E as the total mechanical energy, E = K + U, that we used in PHY181)*

So:

$$E = K + mc^2$$

K = Kinetic Energy, i.e. energy of motion

mc^2 = "Rest Energy." Energy that the mass has all the time – even when it is at rest!

So for a mass m at rest (K = 0):

$$E = mc^2$$

What does it mean?

Mass and energy are the same thing! Mass can be converted into energy and energy can be converted into mass!

*Congratulations! You have derived the most famous equation in Physics – **Get your Sticker!***

However, this is not the way Einstein originally arrived at E = mc^2

Whiteboard Problem 36-14: where's our old friend K = (1/2)mu²?

Show that in the low speed limit (u << c) that we still have our old Newtonian friend for Kinetic Energy:

$$K = \tfrac{1}{2}mu^2$$

We have for the total relativistic energy:

$$E = K + mc^2 = \gamma_p mc^2$$

So:

$$K = (\gamma_p - 1)mc^2$$

$$= \left\{ \frac{1}{\sqrt{1 - u^2/c^2}} - 1 \right\} mc^2$$

$$= \left\{ (1 - u^2/c^2)^{-1/2} - 1 \right\} mc^2$$

Use the binomial expansion:

$$(1+x)^n \approx 1 + nx \text{ for } x << 1$$

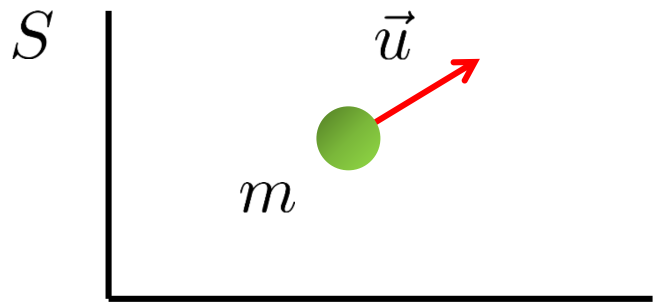
$$\approx \left\{ \left(1 + \frac{u^2}{2c^2}\right) - 1 \right\} mc^2$$

$$\approx \frac{u^2}{2c^2} mc^2 \approx \frac{1}{2}mu^2$$

Once again, Newtonian physics is the low speed (u << c) approximation. (LC)

Dynamics in SR: A Summary

We have developed the **Basic Equations of Relativistic Dynamics.**



Here's a summary:

	Relativistic	Newtonian (u << c)
Momentum	$\vec{p} = \gamma_p m\vec{u}$	$\vec{p} = m\vec{u}$
Kinetic Energy	$K = (\gamma_p - 1)mc^2$	$K = \frac{1}{2}mu^2$
Total Relativistic Energy	$E = K + mc^2 = \gamma_p mc^2$	--------
Relation between Energy and Momentum	$E^2 = p^2c^2 + m^2c^4$	$K = \frac{p^2}{2m}$

Where: $\gamma_p = \frac{1}{\sqrt{1 - u^2/c^2}}$

. . . let's try some really fun and interesting problems now.

Whiteboard Problem: 36-15

An electron is accelerated from rest through a potential difference of 2.0×10^6 Volts.

What is its speed in m/s? (LC)

(Hint: use Conservation of Energy)

Thou Shalt Not Exceed the Speed of Light! Not even by a little bit!

It is commonly believed that Einstein constructed Special Relativity on the idea that nothing can travel faster than the speed of light.

However that idea is not found in the starting point of SR, i.e. the postulates:

1. **The laws of physics (Mechanics and Maxwell's Electricity and Magnetism) are the same in all Inertial Reference Frames.**
2. **The speed of light (in vacuum) is c in all Inertial Frames of Reference**

The ***"Cosmic Speed Limit"*** of c is actually a consequence of these postulates, and is contained in all that follows – i.e. the Lorentz Transformations, time dilation, length contraction, and relativistic dynamics. **In all of these equations is the Lorentz Factor:**

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

For $v \to c \Rightarrow \gamma \to \infty$ and both the momentum and energy go to infinity.

For $v > c \Rightarrow \gamma$ is imaginary and the Lorentz Transformations don't make sense.

Every few years, there's a report about finding something travelling faster than light, but, on closer inspection, the report is refuted. **. . . But someday, who knows.**

(http://en.wikipedia.org/wiki/Faster-than-light_neutrino_anomaly)

Whiteboard Problem: 36-16

We have seen that through ***$E = mc^2$*** that matter and energy are the same thing, and we can convert energy into matter and matter into energy. This is most apparent in nuclear reactions, like those that power the sun.

The nuclear reaction that powers the Sun is the fusion of four protons (i.e. four Hydrogen nuclei) into a Helium nucleus. The process involves several steps, but the **net reaction is simply 4 protons → ^{4}He + energy**. The mass of a Helium nucleus is known to be 6.64 X 10^{-27} kg.

a) **How much energy is released in each fusion? (LC)**
   (Hint: check the initial and final masses, and think $E=mc^2$)

b) A small jug of water has a mass of 1 kg. **Determine how much energy would be released if all of the Hydrogen atoms in the jug were fused to Helium. (LC)**
   How does this compare to the total amount of energy used in the United States in 2010 which was about 1 X 10^{20} J?

Whiteboard Problem: 36-17 *(Your last whiteboard problem!)*

In an attempt to reduce the extraordinarily long travel times for voyaging to distant stars, some people have suggested traveling at close to the speed of light and taking advantage of time dilation. **Suppose you wish to visit the red giant star Betelgeuse, which is 430 ly away, and that you want your 10^8 kg starship to move so fast that you age only 10 years during the trip to Betelgeuse.**

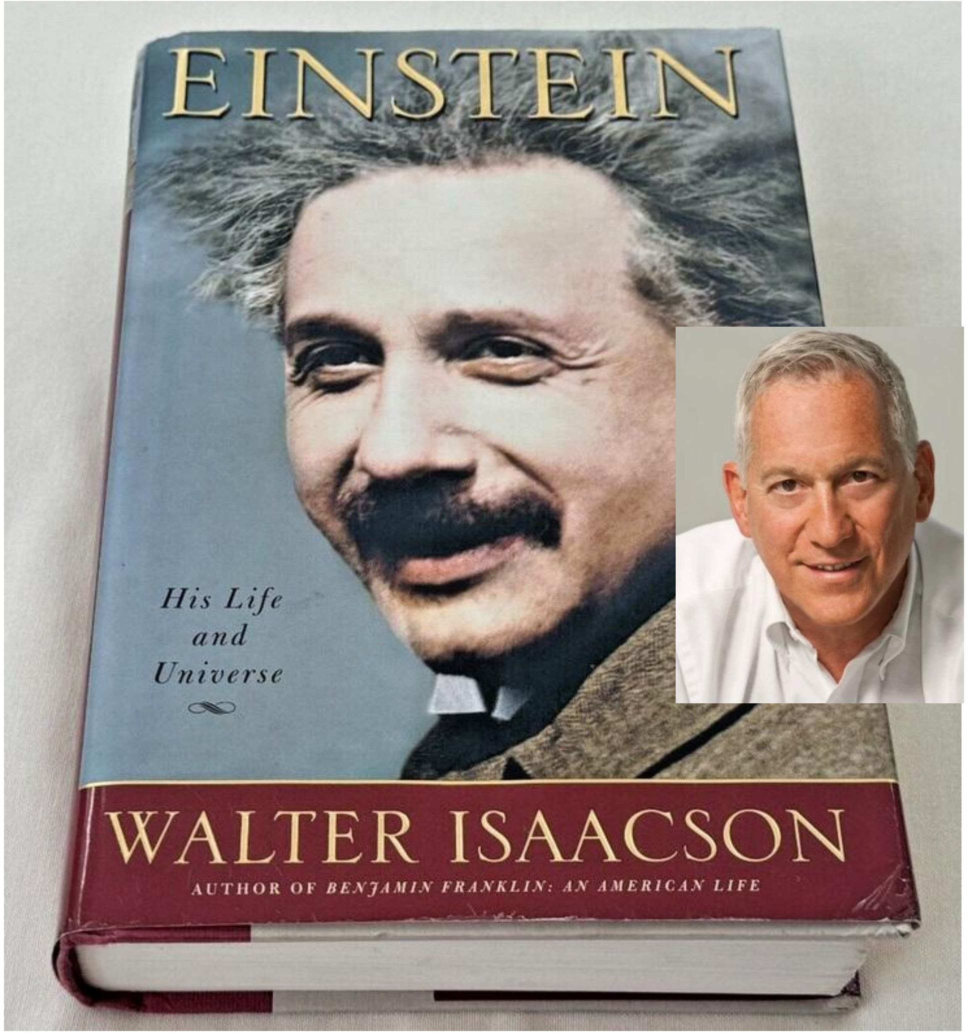
a. **How fast must the starship travel relative to the Earth? (LC)** (ignore the acceleration and deceleration phases)

b. *Ok, part a seems reasonable, but before we all start packing our bags for Betelgeuse, how much is this going to cost?* **How much energy is needed to accelerate the starship to this speed? (LC)** (*Note that the starship is about the size of a fully loaded Nimitz class aircraft carrier. Do you notice anything curious about your answer? Think of the Solar Luminosity.*)

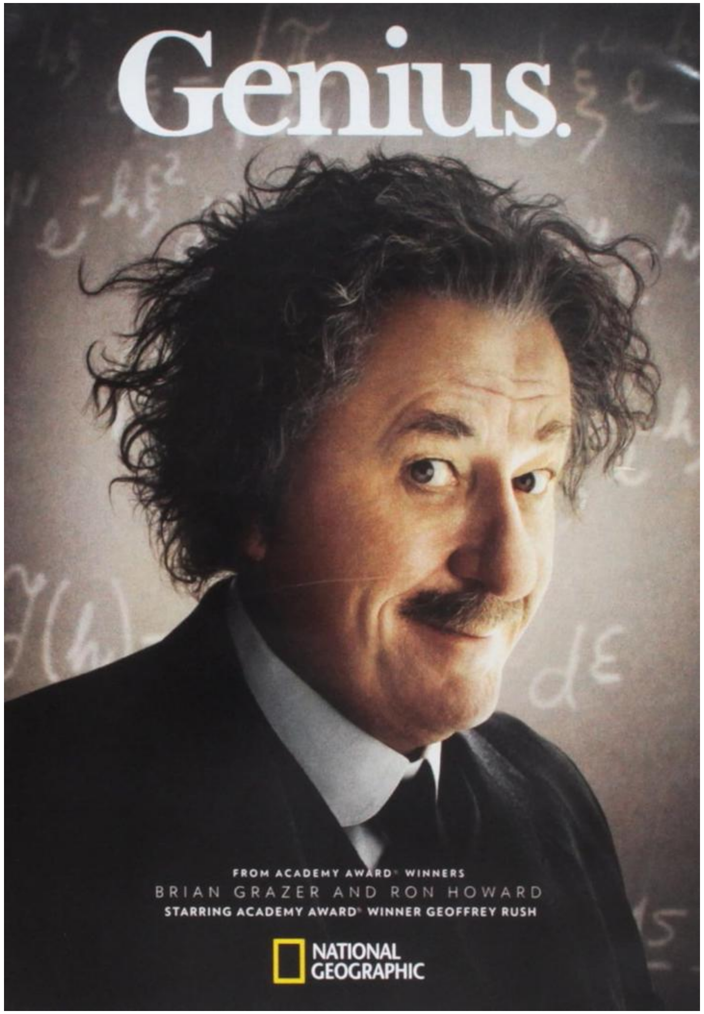
c. Energy is priced by the kilowatt-hour (kWh), and today, the price is about \$0.10/kWh *(Note: 1.0 kWh = 3.6 X 10^6 J).* **How much will it cost in dollars to accelerate the starship to the necessary speed? (LC)**

More on Einstein – if you're interested

A Really Great Biography



A Wonderful 10-Part Documentary



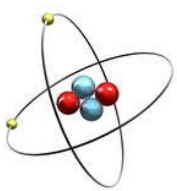
Official Trailer

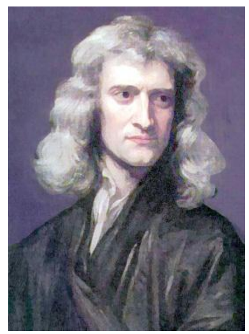
(active link)

Did Einstein Prove Newton Wrong?

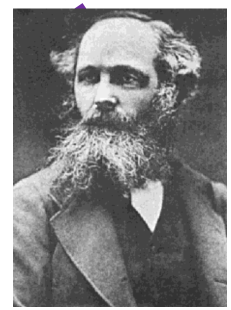
Not really, both Relativity and Quantum Mechanics put bounds on Newtonian and Maxwellian Physics that define where they are an accurate description of reality, but do not apply beyond those boundaries.

Quantum Mechanics



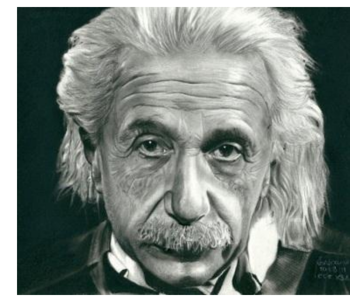


Newtonian Physics



Maxwellian E&M

Special and General Relativity



Someday, someone will discover the bounds for both Relativity and Quantum Mechanics!