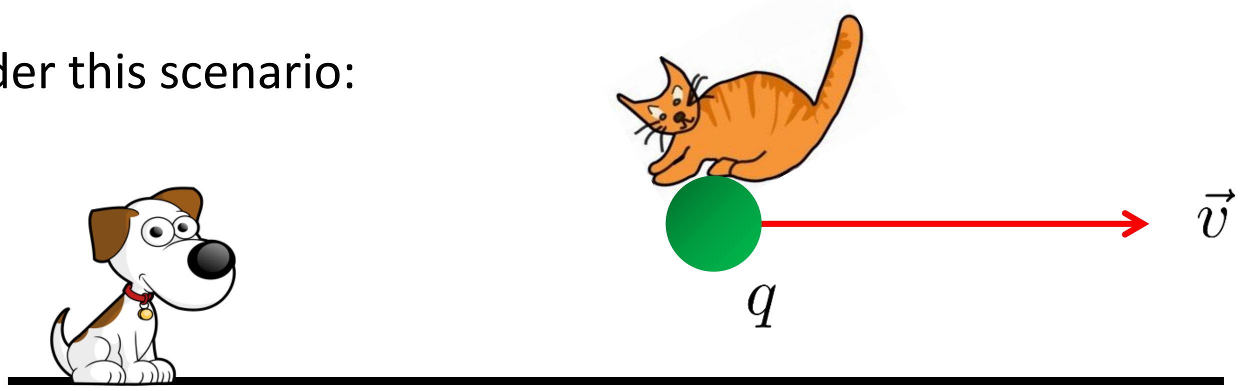
31: The Transformation of Fields

So far in our journey through electromagnetism, we've seen how charges create both electric and magnetic fields; however, if you've been paying **very close attention**, you have perhaps noticed a **very serious problem.**

The Problem: Consider this scenario:



The dog observes a moving charge; so: Dog sees $\underline{\text{Both}}$ $\vec{E}$ and $\vec{B}$

What about a cat that rides on the charge?

The cat observes a stationary charge, so: Cat sees $\underline{\text{Only}}$ $\vec{E}$

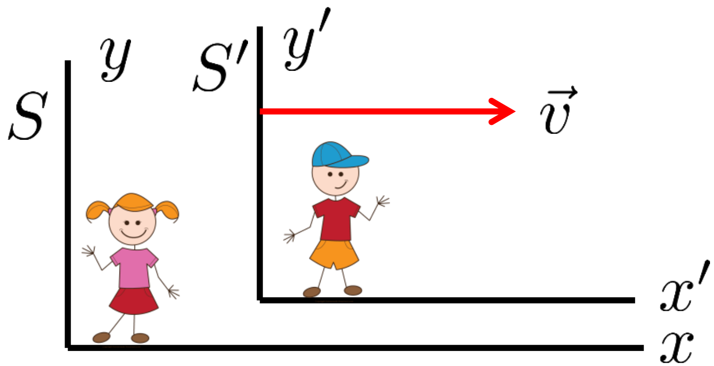
How do we resolve this contradiction?

Both observers – in this case, cat and dog – must observe the same forces and accelerations. But this means that the observed fields are relative.

The Transformation of Fields

Your author goes through a lengthy (*and confusing*) derivation of how to transform electric and magnetic fields. He imposes the fact that the two observers must agree on any acceleration caused by the fields. This then tells us how the observed fields change.

In the language of Special Relativity*, we have **two frames of reference, S and S', where S' is moving relative to S:**



Observers in S measure: $\vec{E}$ and $\vec{B}$

Observers in S' measure: $\vec{E}'$ and $\vec{B}'$

Note: the prime here has nothing do with differentiation; it just means that S' is a different frame than S, etc.

If we know the fields in frame S:

$$\vec{E}' = \vec{E} + \vec{v} \times \vec{B}$$
$$\vec{B}' = \vec{B} - \epsilon_0 \mu_o \vec{v} \times \vec{E}$$

If we know the fields in frame S':

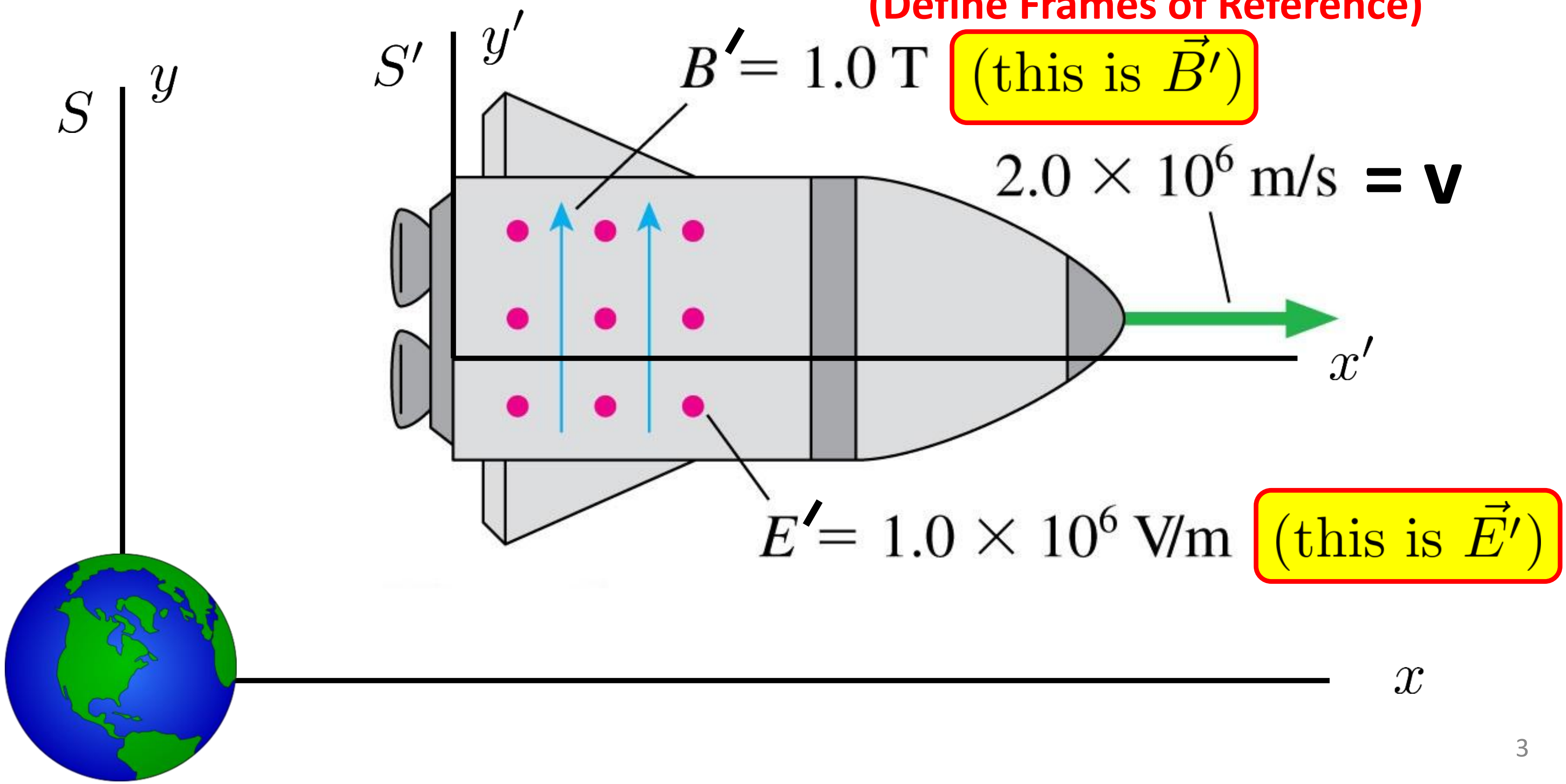
$$\vec{E} = \vec{E}' - \vec{v} \times \vec{B}'$$
$$\vec{B} = \vec{B}' + \epsilon_0 \mu_o \vec{v} \times \vec{E}'$$

These relations say that there is really only one field, the electromagnetic field. The distinction between electric and magnetic is relative. *Note also, these transformations are nonrelativistic; i.e. they work only for v << c.

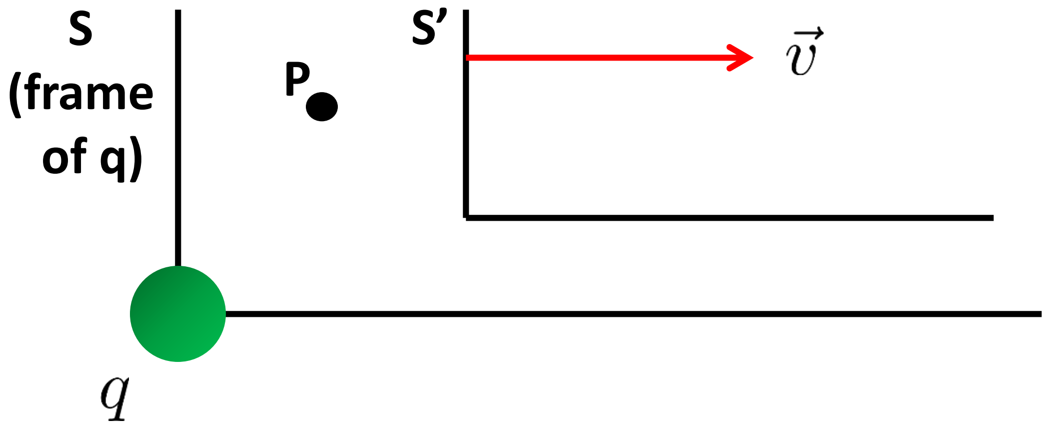
Whiteboard Problem: 31-1

A rocket zooms past earth at v = 2 X 10^6 m/s. Scientists on the rocket have created the electric and magnetic fields shown in the figure below. **What fields, in component form, are measured by an earthbound scientist?** (Enter E in LC)

(Define Frames of Reference)



Where does the Biot-Savart Law Come From?



In frame S, observers measure at point P: $\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$ $\quad \vec{B} = 0$

Observers in S' observe an electric field **and a magnetic field:**

$$\vec{B}' = \vec{B} - \epsilon_0\mu_o\vec{v}\times\vec{E} \quad = -\epsilon_0\mu_0\vec{v}\times\left(\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}\right) \quad = -\frac{\mu_0}{4\pi}\frac{q\vec{v}\times\hat{r}}{r^2}$$

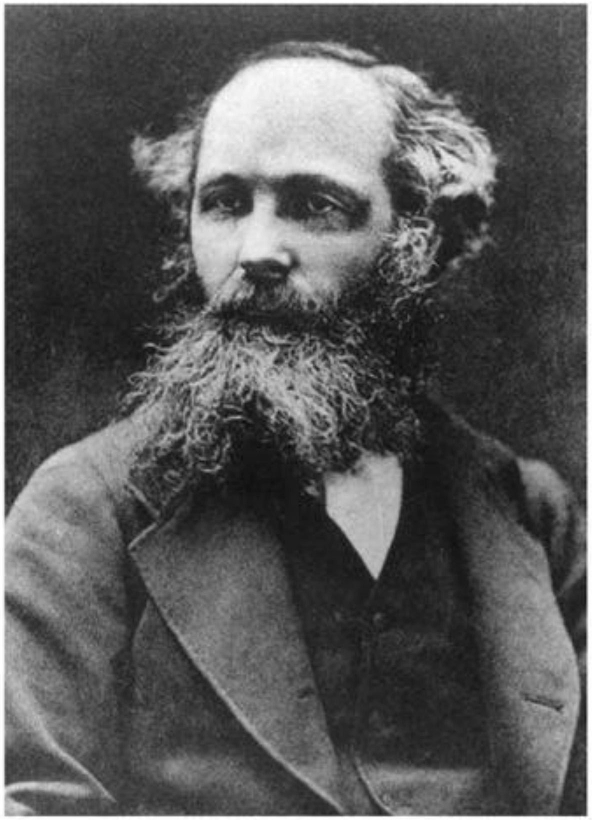
And in frame S', the charge has velocity: $\vec{v'} = -\vec{v}$

$$\vec{B}' = \frac{\mu_0}{4\pi}\frac{q\vec{v'}\times\hat{r}}{r^2}$$ **The Biot-Savart Law!**

So, the Biot-Savart Law is just the transformed Coulomb electric field!

Maxwell and the Equations

In the 1860's, the Scottish physicist, **James Clerk Maxwell** brought together everything that had been done with electricity and magnetism by Gauss, Ampere, Faraday, and others. **He adopted the field concept from Faraday**, and put everything into a consistent mathematical framework, similar to, but not quite, what we use today. (*The full mathematics of vectors that we use today didn't exist until after Maxwell*.)



Although he didn't discover all of these laws, we call them **Maxwell's Equations**.

Albert Einstein on James Clerk Maxwell and The Idea of Fields

This change in the conception of reality is the most profound and the most fruitful that physics has experienced since the time of Newton. Refering to James Clerk Maxwell's contributions to physics.

- Albert Einstein -

The Field Equations . . . almost Maxwell's Equations

Gauss' Law for the Electric Field	$\oint_S \vec{E} \cdot d\vec{A} = \frac{Q_{\text{in}}}{\epsilon_0}$	Charges create $\vec{E}$
Gauss' Law for the Magnetic Field	$\oint_S \vec{B} \cdot d\vec{A} = 0$	magnetic monopoles don't exist, $\vec{B}$ forms closed loops
Ampere's Law	$\oint \vec{B} \cdot d\vec{s} = \mu_0 I_{\text{thru}}$	Currents create $\vec{B}$
Faraday's Law	$\oint \vec{E} \cdot d\vec{s} = -\frac{d}{dt} \int \vec{B} \cdot d\vec{A}$	A Changing $\vec{B}$ creates $\vec{E}$

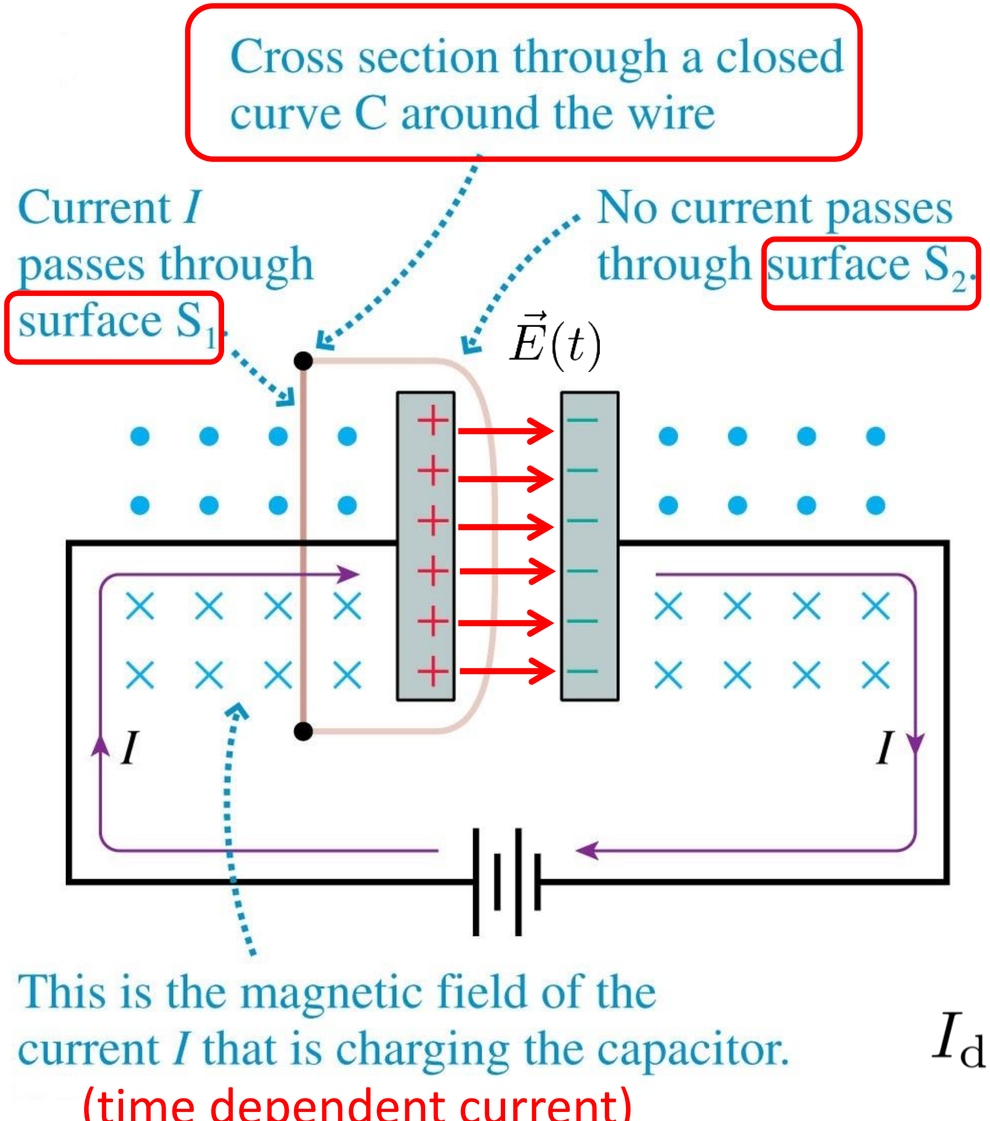
Do you see a problem with these equations?

. . . Maxwell did:

There are two ways to create an electric field, but only one way to create a magnetic field. Maxwell believed there should be symmetry. **Faraday's Law says a changing magnetic field creates an electric field; is the reverse true?**

Maxwell's Displacement Current

Maxwell was able to fill in the missing piece by considering a **Charging Capacitor**:



If you apply Ampere's Law to **Curve C** and Surface S_1, you would say there is a magnetic field on **C** since current goes through S_1.

But, if you apply Ampere's Law to **Curve C** and Surface S_2, you would say there is no magnetic field on **C** since no current goes through S_2.

But, any fool can put a compass on C and see there is a magnetic field!

What does go through surface S_2?

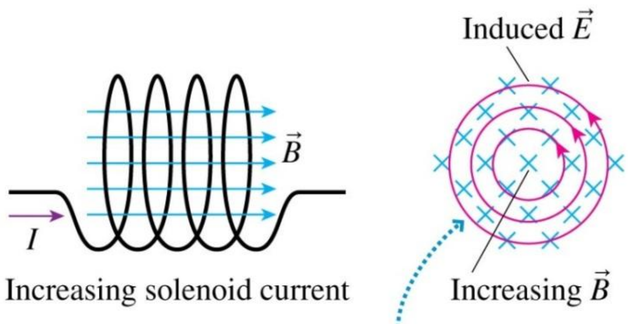
A time-varying Electric Field.

Maxwell proposed that this would act in Ampere's Law like a current,
he called it "The Displacement Current":

$$I_{\text{disp}} = \epsilon_0 \frac{d\Phi_e}{dt} = \epsilon_0 \frac{d}{dt} \int \vec{E} \cdot d\vec{A}$$

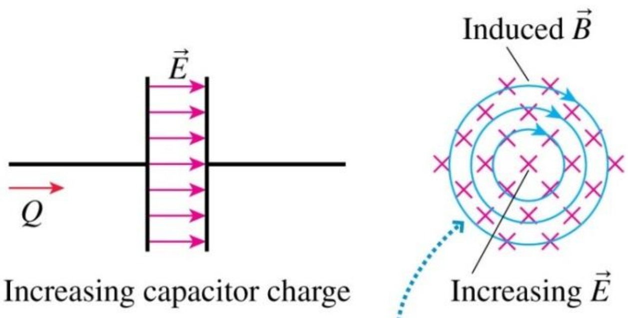
Whiteboard Problem: 31-2

A 5.0 cm diameter parallel plate capacitor has a 0.5 mm gap between the plates. **Find the displacement current in the capacitor if the potential difference across the capacitor is increasing at 500,000 V/s.** **(LC)**



Faraday's law describes an induced electric field.

This is from Chapter 30. An increasing B in the solenoid creates a circulating E.



The Ampère-Maxwell law describes an induced magnetic field.

This is what is going on in the this problem. An increasing E creates a circulating B. You're to find the **displacement current** – which is not an actual current:

$$I_{\text{disp}} = \epsilon_0 \frac{d\Phi_e}{dt} = \epsilon_0 \frac{d}{dt} \int \vec{E} \cdot d\vec{A}$$

Finally . . . Maxwell's Equations

Gauss' Law for the Electric Field	$$\oint_S \vec{E} \cdot d\vec{A} = \frac{Q_{\text{in}}}{\epsilon_0}$$	Charges create $\vec{E}$
Gauss' Law for the Magnetic Field	$$\oint_S \vec{B} \cdot d\vec{A} = 0$$	magnetic monopoles don't exist, $\vec{B}$ forms closed loops
Ampere-Maxwell Law	$$\oint \vec{B} \cdot d\vec{s} = \mu_0 \left(I_{\text{thru}} + \epsilon_0 \frac{d}{dt} \int \vec{E} \cdot d\vec{A} \right)$$	Currents and a changing $\vec{E}$ create $\vec{B}$
Faraday's Law	$$\oint \vec{E} \cdot d\vec{s} = -\frac{d}{dt} \int \vec{B} \cdot d\vec{A}$$	A Changing $\vec{B}$ creates $\vec{E}$

Behold Maxwell's Equations! Combined with the **Lorentz Force Law** $(\vec{F} = q\vec{E} + q\vec{v} \times \vec{B})$ these give a complete description of all (non quantum) electromagnetic phenomena.

Along with Newton's Laws and the Laws of Thermodynamics, this constitutes what we call Classical Physics.

Electromagnetic Waves

Your author shows that a sinusoidal wave in the electric and magnetic fields satisfies the source free Maxwell's equations. *This is a very tedious and un-illuminating process using the integral form of Maxwell's equations.*

With a little more math – *things that you'll learn in Calculus III* – you can start with Maxwell's equations and show that electromagnetic waves exist. **This is what Maxwell did**.

Put Your Pens and Pencils Down! This won't be on any homework or exams!

The Differential Forms of Maxwell's Equations:

Gauss' Law for the Electric Field: $\oint_S \vec{E} \cdot d\vec{A} = \frac{Q_{\text{in}}}{\epsilon_0} \Leftrightarrow \nabla \cdot \vec{E} \overset{*}{=} \frac{\rho}{\epsilon_0}$

Gauss' Law for the Magnetic Field: $\oint_S \vec{B} \cdot d\vec{A} = 0 \Leftrightarrow \nabla \cdot \vec{B} = 0$

Ampere-Maxwell Law: $\oint \vec{B} \cdot d\vec{s} = \mu_0 \left(I_{\text{thru}} + \epsilon_0 \frac{d}{dt} \int \vec{E} \cdot d\vec{A} \right) \Leftrightarrow \nabla \times \vec{B} \overset{*}{=} \mu_0 \vec{J} + \mu_o \epsilon_0 \frac{\partial \vec{E}}{\partial t}$

Faraday's Law: $\oint \vec{E} \cdot d\vec{s} = -\frac{d}{dt} \int \vec{B} \cdot d\vec{A} \Leftrightarrow \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

*This is called the divergence, it is a measure of how much a vector field diverges from a point.

*This is called the curl; it is a measure of how much a vector field goes around a point.

Electromagnetic Waves

Maxwell showed that if you start with the source free form of the equations:

$$\rho = \vec{J} = 0$$

i.e. empty space with no charge and no currents, the fields behave as:

$$\nabla^2 \vec{E} = \epsilon_0 \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2} \qquad \nabla^2 \vec{B} = \epsilon_0 \mu_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

What are these? Perhaps if we look at just the x-component of the electric field equation:

$$\frac{\partial^2 E_x}{\partial x^2} = \underbrace{\epsilon_0 \mu_0}\frac{\partial^2 E_x}{\partial t^2}$$

What is this?

This is a wave equation!

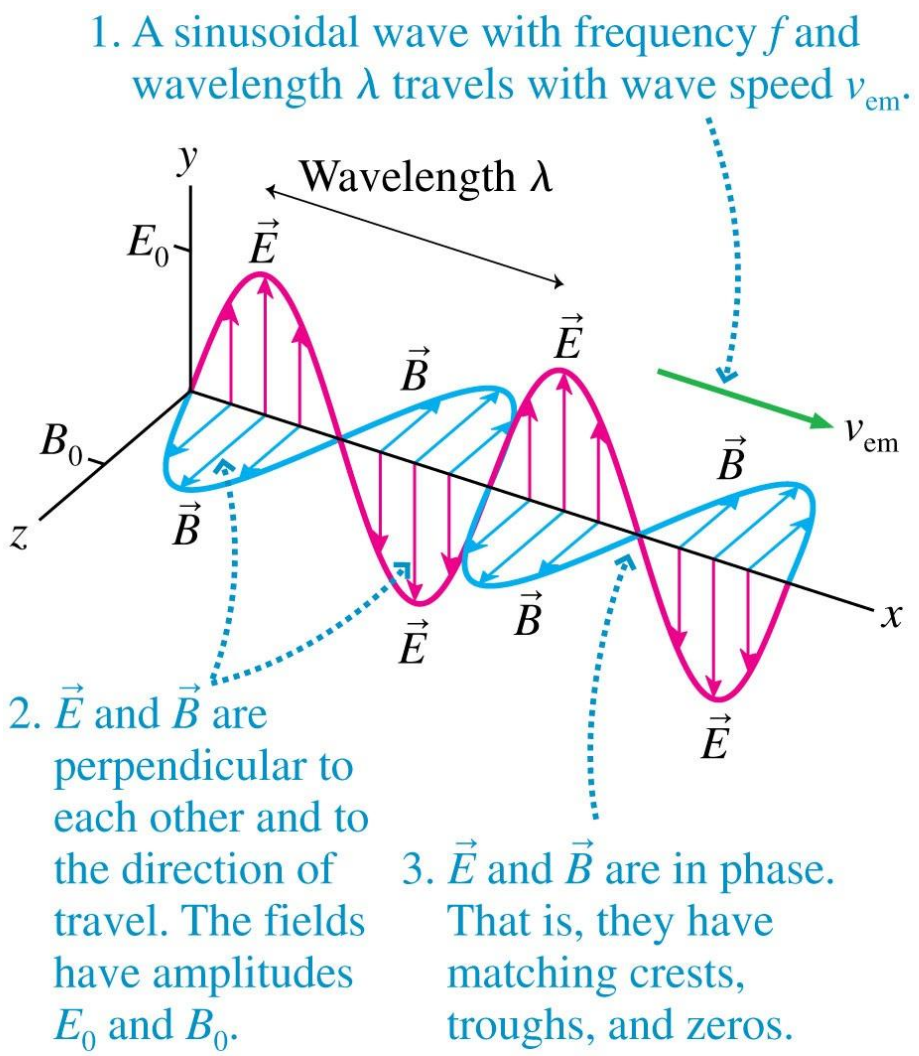
One over the square root of whatever is in here is the wave speed.

$$\text{So, wave speed} = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = \text{speed of light, } c \approx 3 \times 10^8 \ m/s$$

(try it with your calculator)

Electromagnetic Waves

One solution to the electromagnetic wave equation is a sinusoidal travelling wave:



A changing E creates a changing B, and, in turn, a changing B creates a changing E. The whole disturbance then propagates in space.

And we saw an account in **The Story of Electricity Part 3** of how this was experimentally verified in the experiments of Heinrich Hertz.

Notice also that $\vec{E} \times \vec{B}$ always gives the direction of propagation.

Optional Arvin Ash Video

The physicist Arvin Ash produces short videos on Youtube about many topics in physics.
Most of these videos are excellent, and they're usually only about 15 minutes long.
We'll be posting links for some these videos for the next few classes. These are optional, but they should help in understanding the material that we'll be covering in class.

Arvin Ash on Maxwell's Equations

In this episode, Ash describes the history of Maxwell and his equations. He shows the differential forms of the equations, but he doesn't do any of the mathematical manipulations – his descriptions are mostly qualitative.

Why is the Speed of Light What it is?

(This is an active link that will take you to Youtube. if the link doesn't work, copy this URL into your browser: https://youtu.be/FSEJ4YLXtt8)