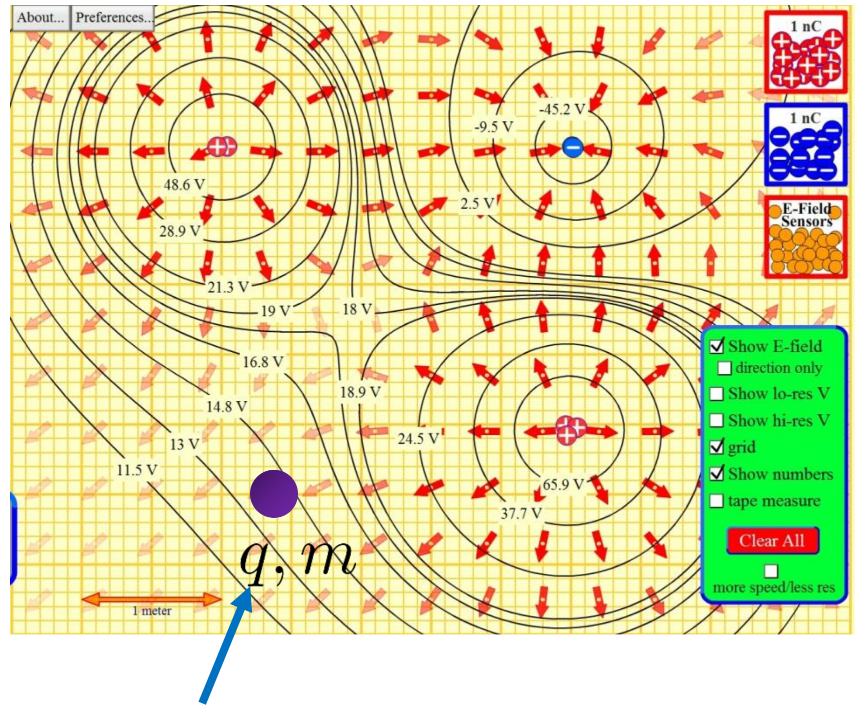
26: Connecting the Electric Field and the Potential

Remember what the **Electric Field** and **Potential** are:

They both describe how source charges communicate their electric effect over all space – see the PhET image



$\vec{E}$ associates a **vector** with each point in space

V associates a **scalar** with each point in space

They both describe the same thing, but in a slightly different language.
What's the difference?

Suppose we released a particle with **charge q and mass m** at some point. **How would we find its subsequent motion?**

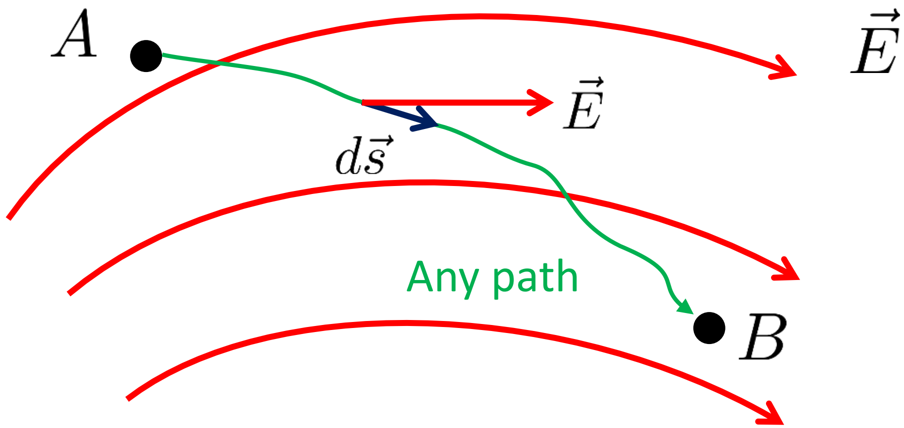
Using the Field: $\vec{F} = q\vec{E} \Rightarrow$ find the motion with Newton's 2^{nd} Law, $\vec{F} = m\vec{a}$

Using the Potential: $U = qV \Rightarrow$ find the motion with Energy Conservation, $\Delta E = \Delta K + \Delta U$

How can we go back and forth between these two descriptions?

Going from the Electric Field to the Potential

i.e. If we know $\vec{E}$ in a region of space, how do we find V?



What is the potential difference between points A and B?

Recall from PHY181, the **Potential Energy** associated with a **Conservative Force** is:

$$\Delta U = -\int \vec{F} \cdot d\vec{s}$$ (*i.e.* minus the work done by F)

And for the **Electric Force**: $\vec{F} = q\vec{E}$ and $\Delta U = q\Delta V$

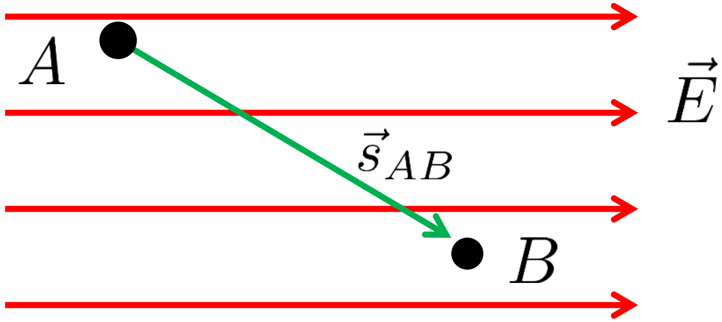
So: the potential difference between two points is:

$$\Delta V = V_B - V_A = -\int_A^B \vec{E} \cdot d\vec{s}$$

Where the integral can be done over any path between A and B, since the electric force is conservative.

Going from the Electric Field to the Potential

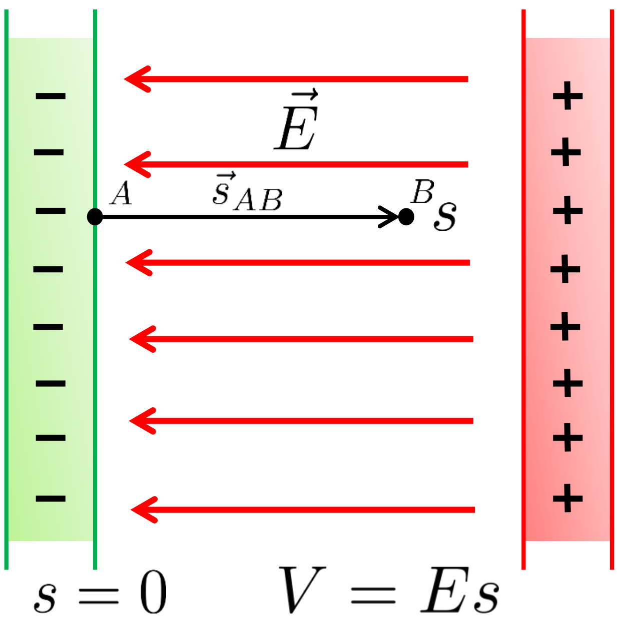
For a Uniform Field



$$\Delta V = V_B - V_A = -\int_A^B \vec{E} \cdot d\vec{s}$$

$$= -\vec{E} \cdot \int_A^B d\vec{s} \quad = -\vec{E} \cdot \vec{s}_{AB}$$

Is this consistent with what we did last class?



$$\Delta V = V_B - V_A = -\vec{E} \cdot \vec{s}_{AB}$$

$$= -Es\cos 180^\circ$$

$$= Es$$

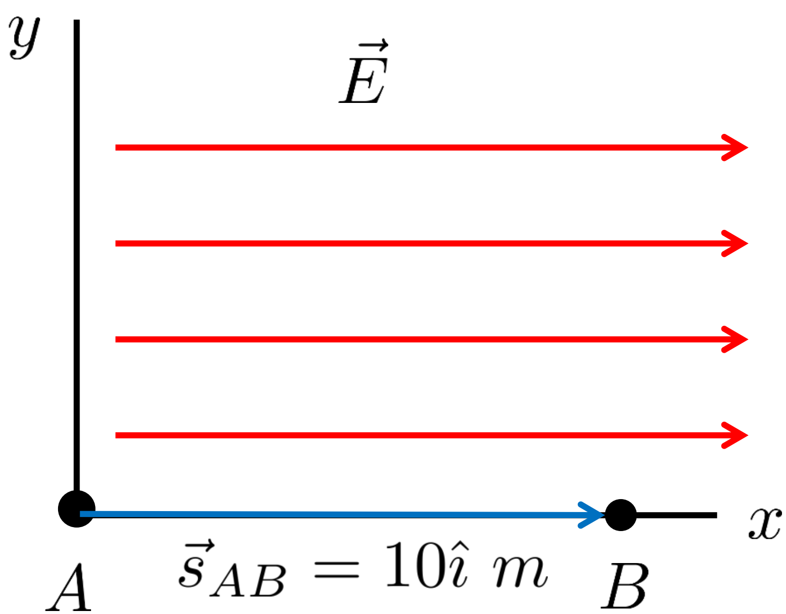
Perhaps you learned in a previous course that the ***Potential is the field times the distance***, **i.e. V = Ed.** This is a special case. **The following whiteboard problems show cases where V = Ed isn't always right.**

Whiteboard Problem 26-1: Potential from Field 1 (LC)

Find the potential difference, $\Delta V = V_B - V_A = -\int_A^B \vec{E} \cdot d\vec{s}$, between the origin, $(x = 0, y = 0)\ m$, and point B at $(x = 10, y = 0)\ m$ for the electric field:

$$\vec{E} = 750\hat{\imath}\ V/m$$

First: Draw it:



$$\Delta V = V_B - V_A = -\int_A^B \vec{E} \cdot d\vec{s}$$

$$= -\vec{E} \cdot \int_A^B d\vec{s} \quad = -\vec{E} \cdot \vec{s}_{AB}$$

$$= -(750\ V/m)(10\ m)$$

$$= -7500\ V$$

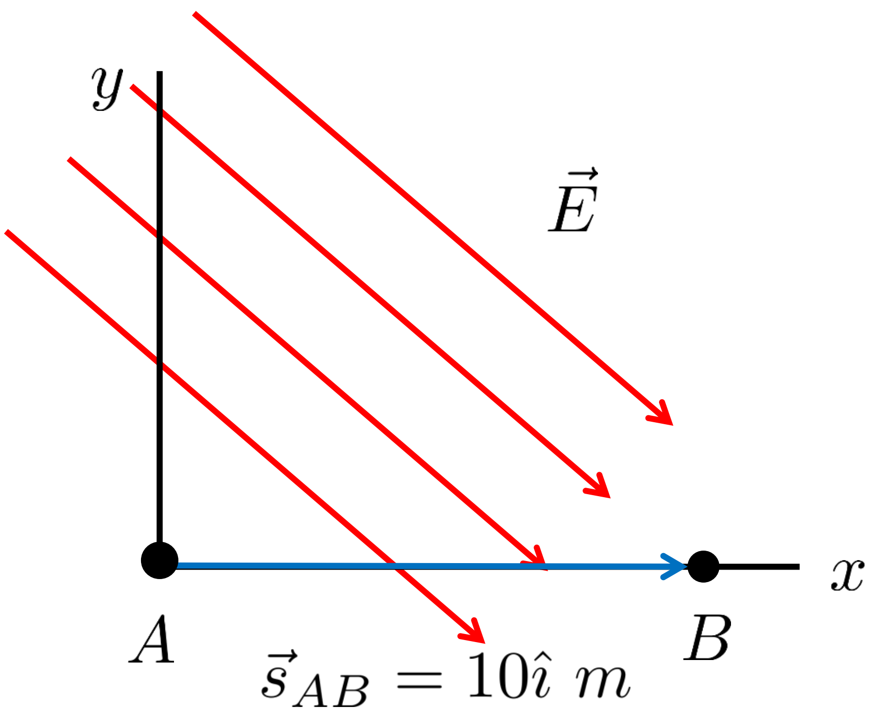
Note: V = Ed gives +7500 V; i.e. it doesn't get the sign right. For this field, $V_B < V_A$.

Whiteboard Problem 26-1: Potential from Field 2 (LC)

Find the potential difference, $\Delta V = V_B - V_A = -\int_A^B \vec{E} \cdot d\vec{s}$, between the origin, $(x = 0, y = 0)\ m$, and point B at $(x = 10, y = 0)\ m$ for the electric field:

$$\vec{E} = 750\hat{\imath} - 600\hat{\jmath}\ V/m$$

First: Draw it:



$$\Delta V = V_B - V_A = -\int_A^B \vec{E} \cdot d\vec{s}$$

$$= -\vec{E} \cdot \int_A^B d\vec{s} \quad = -\vec{E} \cdot \vec{s}_{AB}$$

$$= -(750\ V/m)(10\ m)$$

$$= -7500\ V$$

(*i.e*. only that part of the displacement parallel to the field matters, the same as work done by a force.)

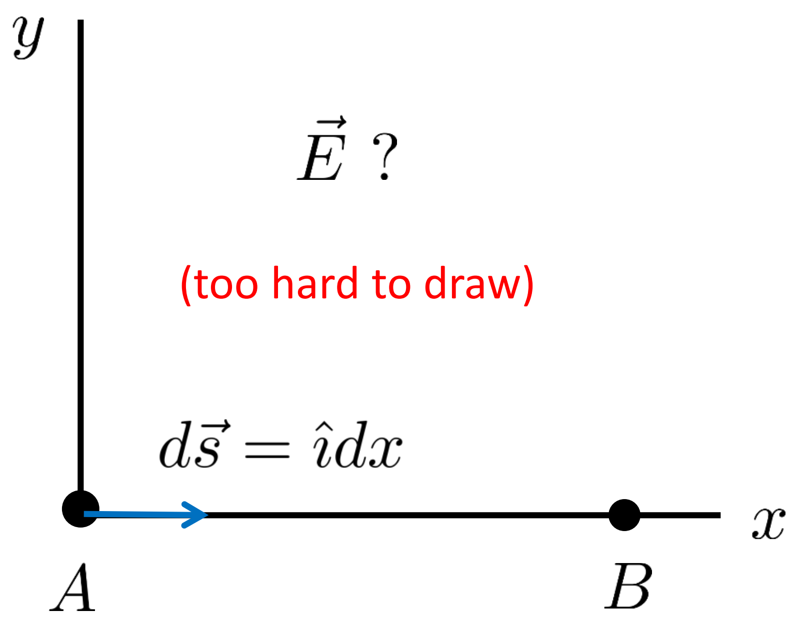
Note: V = Ed gives +9600 V.

Whiteboard Problem 26-1: Potential from Field 3 (LC)

Find the potential difference, $\Delta V = V_B - V_A = -\int_A^B \vec{E} \cdot d\vec{s}$, between the origin, $(x = 0, y = 0)\ m$, and point B at $(x = 10, y = 0)\ m$ for the electric field:

$$\vec{E} = 750x\hat{\imath} - 600y\hat{\jmath}\ V/m$$

First: Draw it:



$$\Delta V = V_B - V_A = -\int_A^B \vec{E} \cdot d\vec{s}$$

$$\vec{E} \cdot d\vec{s} = (750x)(dx)$$

$$\text{So, } \Delta V = -\int_0^{10} 750x\,dx$$

$$= -750\left[\frac{x^2}{2}\right]_0^{10}$$

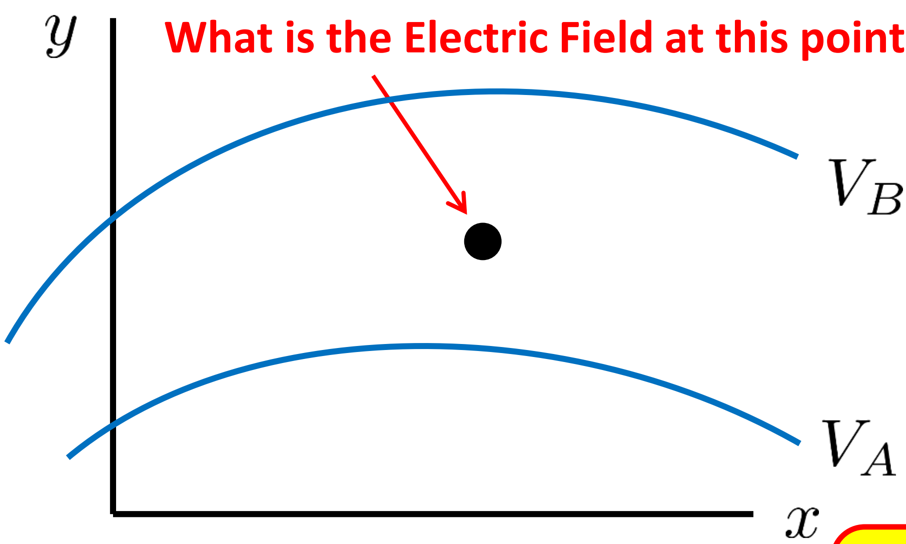
$$\text{So: } \Delta V = V_B - V_A = -37{,}500\ V$$

Note: I don't know what V = Ed gives here!

Going from the Potential to the Electric Field

$i.e.$ If we know V in a region of space, how do we find $\vec{E}$?

What does it mean to know the potential in a region of space? **Know the equipotentials.**



Recall from PHY181 for a force and potential energy (in 1D):

$$F_x = -\frac{dU}{dx}$$

$$\text{and } \vec{F} = q\vec{E} \qquad U = qV$$

$$\text{So: } E_x = -\frac{dV}{dx}$$

In two dimensions, this becomes:

$$E_x = -\frac{\partial V}{\partial x} \quad \& \quad E_y = -\frac{\partial V}{\partial y}$$

Partial Derivatives?

Meaning:

A changing $V \Rightarrow$ there is an $\vec{E}$

$\vec{E}$ points in the direction of decreasing V

$\vec{E}$ is everywhere $\perp$ to equipotential surfaces

A Quick Lesson in Partial Derivatives

- If you have taken Calc III, then you're familiar with partial derivatives, but if you haven't, here's a quick lesson – *it's one of the easiest things in Calculus*.
- In Calc I, you work with functions of a single variable, e.g. f(x), and the derivative of f(x) is the rate of change of f as x changes – called an **ordinary derivative**:

$$\text{So for } f(x)\text{, the derivative, } \frac{df(x)}{dx} = \text{rate of change of } f \text{ as } x \text{ changes}$$

- In Calc III, you'll work with functions of more than one variable, e.g. g(x,y). How do we handle derivatives for a function like this – **partial derivatives**.

$$\text{So for } g(x,y)\text{, the partial derivative, } \frac{\partial g(x,y)}{\partial x} = \text{rate of change of } g \text{ as } x \text{ changes}$$

$$\text{and, the partial derivative, } \frac{\partial g(x,y)}{\partial y} = \text{rate of change of } g \text{ as } y \text{ changes}$$

- Note, the notation: a regular "*d*" for ordinary derivative, a "*curly-cue d*" for the partial derivative.
- **Now, when taking partial derivative with respect to x, what do you do with y? Treat it as a constant! And for the partial with respect to y, you treat x as a constant.**
- Here's a quick example: suppose

$$g(x,y) = 6x^2y + 2xy^3$$

$$\frac{\partial g}{\partial x} = 12xy + 2y^3 \quad \text{and} \quad \frac{\partial g}{\partial y} = 6x^2 + 6xy^2$$

Whiteboard Problem: 26-2

The electric potential in a region of space is

$$V(x,y) = (150x^2 - 200y^2) \text{ Volts}$$

where x and y are in meters.

What is the electric field in component form at (x,y) = (2.0m, 2.0m)? (LC)

Remember: Instructions for entering a vector in component form on LC:

For the vector:

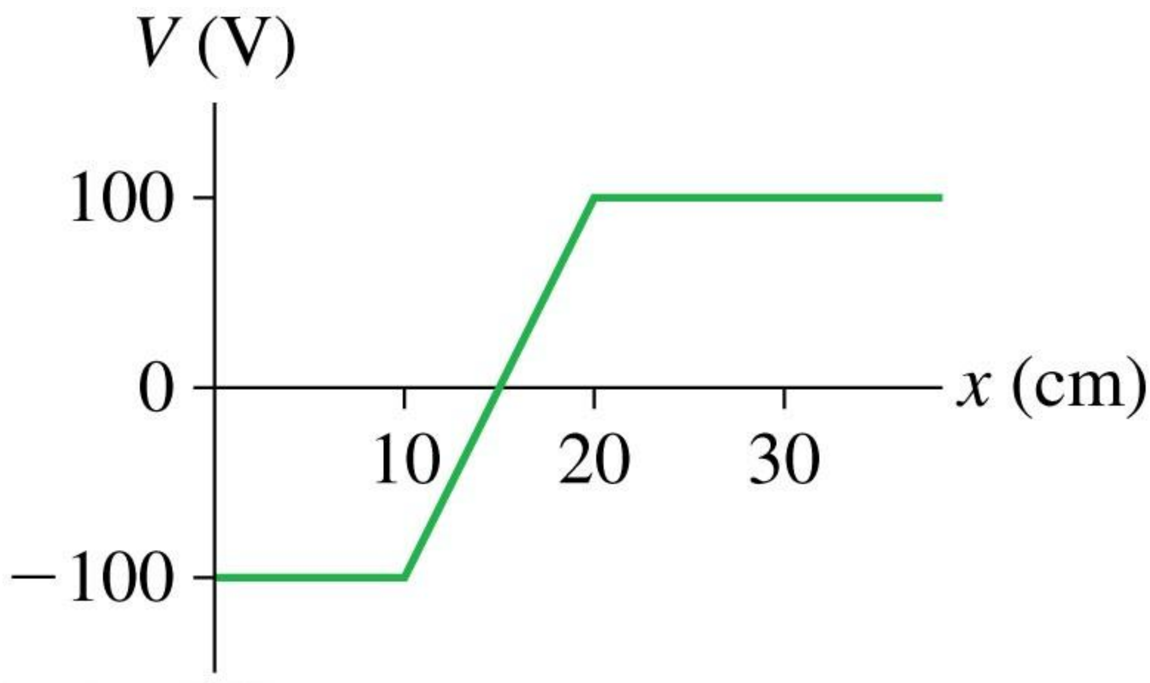
$$\vec{E} = -5.32 \times 10^4 \hat{\imath} + 1.78 \times 10^3 \hat{\jmath}$$

Use only two significant figures and the pull down menu for powers of ten if necessary; put the components in parentheses, and use just i and j for unit vectors – no hats!

For the vector above, enter: **$(-5.3 \cdot 10^4)$i + $(1.8 \cdot 10^3)$j**

Whiteboard Problem: 26-3

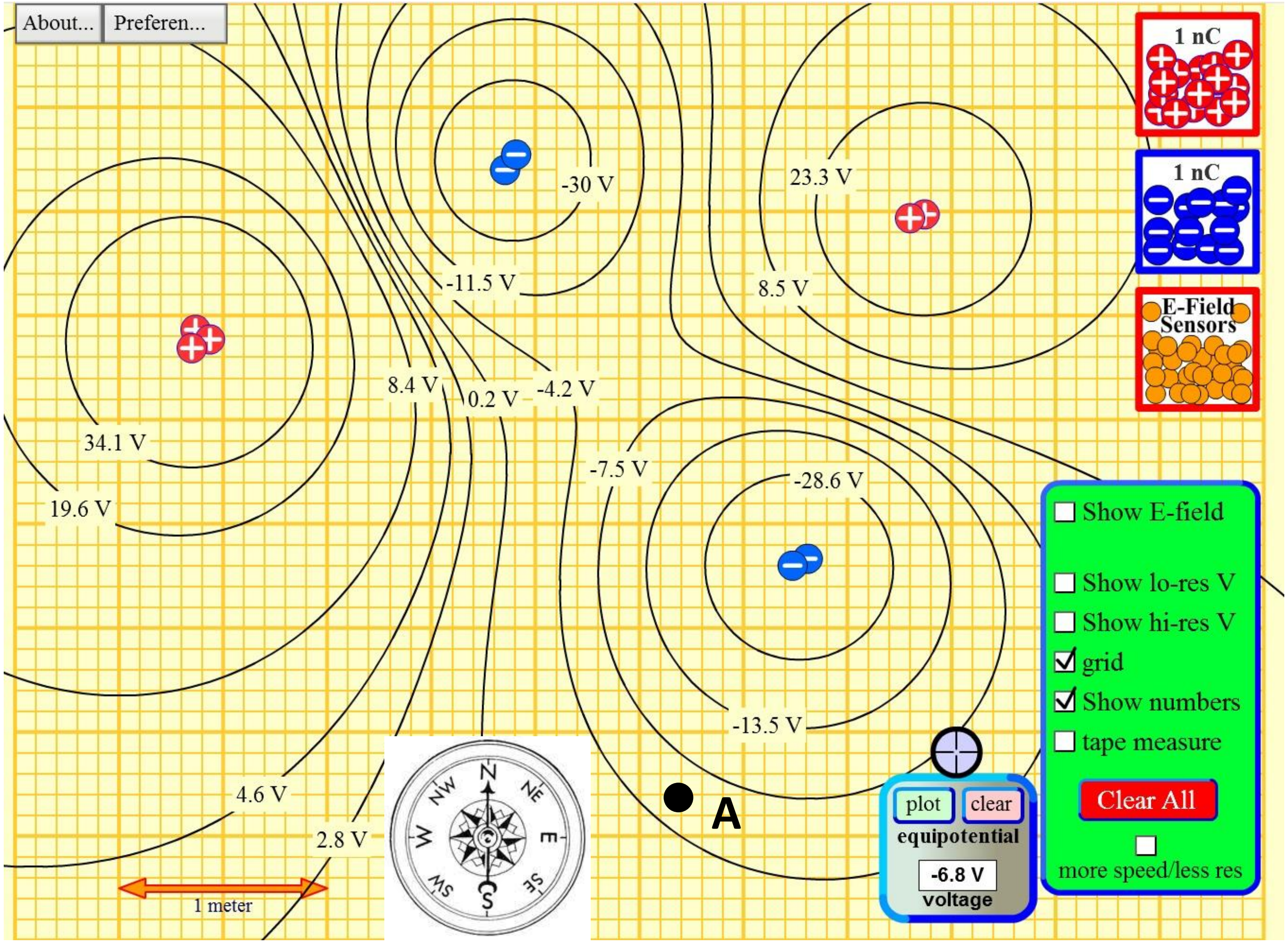
The figure below shows a graph of V versus x.
Draw the corresponding graph of E_x versus x. (LC)



Whiteboard Problem: 26-4

The figure shows the equipotentials for a group of charges.

Find an estimate for the strength of the electric field and its direction at the point labelled A. (LC) (give direction as compass direction – *see the compass*?)



Using the Potential to find the Electric Field

As we saw in Chapter 25, in most cases, it's easier to find the potential for a given distribution of charge. If that is the case, **then it may be easier to get the field from the potential, instead of integrating directly for the field.**

e.g.

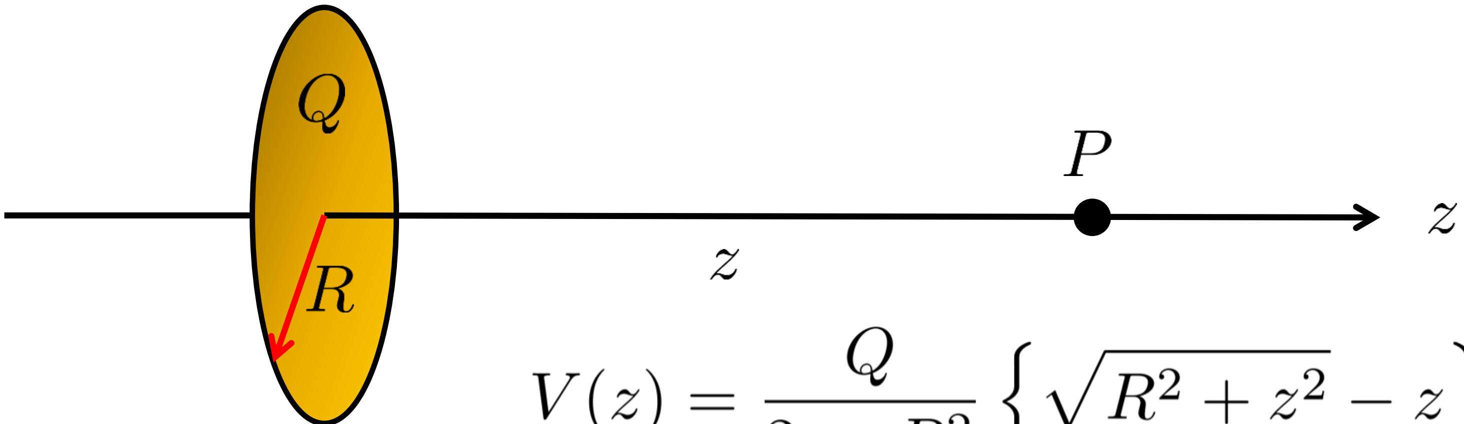
Suppose we know V(x,y) <u>in the region</u> around the point P.

Then, we can get the electric field at P by:

$$E_x = -\left.\frac{\partial V}{\partial x}\right|_{\text{at P}} \quad \& \quad E_y = -\left.\frac{\partial V}{\partial y}\right|_{\text{at P}}$$

Whiteboard Problem: 26-5

In your text, it was found in Example 25-11 that the potential on the axis of a charged disk is:



$$V(z) = \frac{Q}{2\pi\epsilon_0 R^2}\left\{\sqrt{R^2+z^2} - z\right\}$$

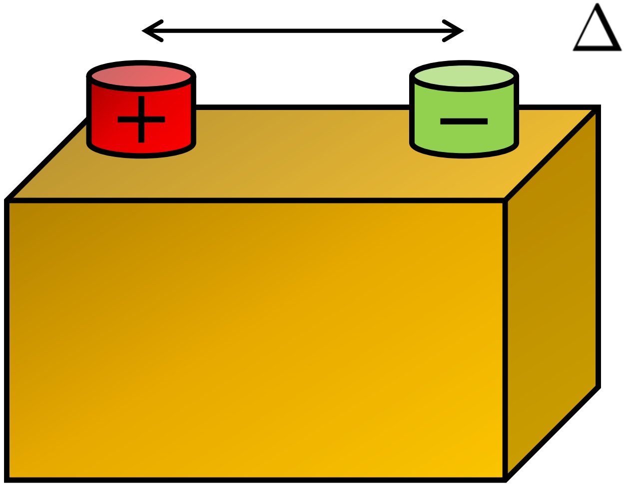
Use this expression for the potential to find an expression for the z-component of the electric field on the axis. (LC)

Sources of Potential (EMF)

A source of potential difference is sometimes called *a* ***source of EMF***, a widely used term, which stands for ***ElectroMotive Force***. Your author points out that this is an outdated term since Force may or may not be involved when there is a potential difference – but you still see it used frequently.

Most sources of potential (or emf) create a potential difference by separating charge either mechanically (e.g. rubbing a rod or a Van de Graff generator) or chemically as in a **battery**:

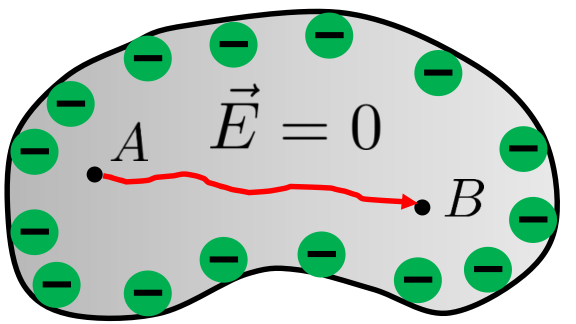
$$\Delta V = V_+ - V_- \;\; (e.g.\;\; 12 \text{ V})$$



A lot of interesting chemistry goes on inside a battery, but for our purposes, we can just consider it to be a **source of constant potential difference between the battery terminals**. (here's an optional video that details the chemistry in a zinc copper battery: https://www.youtube.com/watch?v=7b34XYgADlM&t=791s)

Revisit: The Charged Conductor in Equilibrium

In the Chapter 24 **Readings**, we saw that a conductor in equilibrium has **zero electric field** inside, and any excess charge resides on the exterior surface:

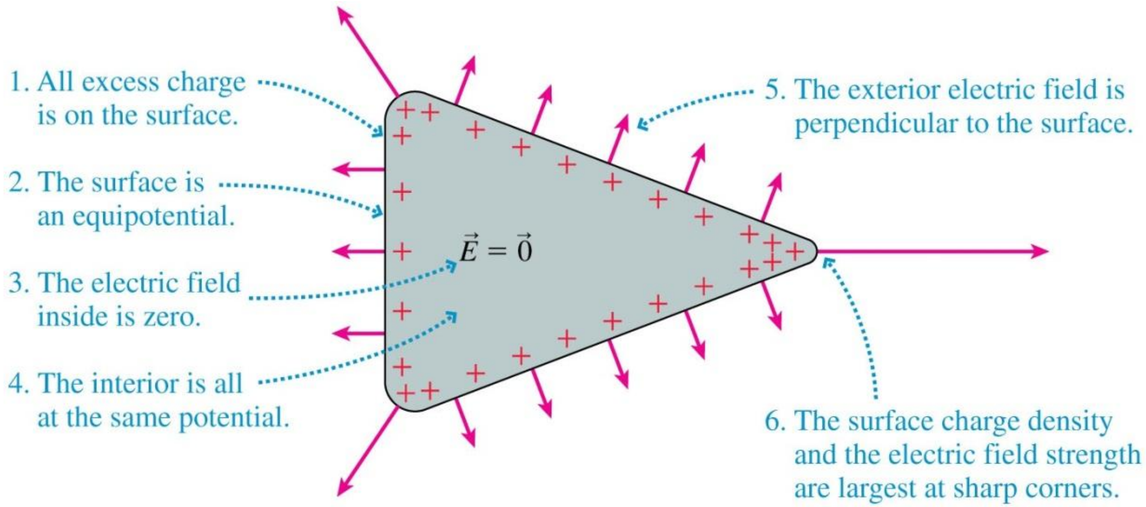


For any two points inside the conductor:

$$\Delta V = V_B - V_A = -\int_A^B \vec{E} \cdot d\vec{s} = 0 \text{ since } \vec{E} = 0 \text{ at all points on any path}$$

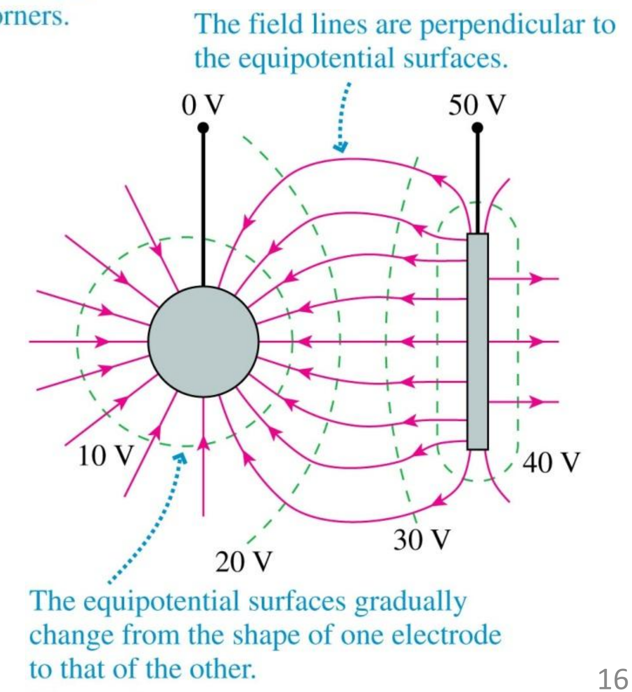
Therefore, all points inside a conductor in equilibrium are at the same potential, and the surface is an equipotential surface.

Summary: The Charged Conductor in Equilibrium



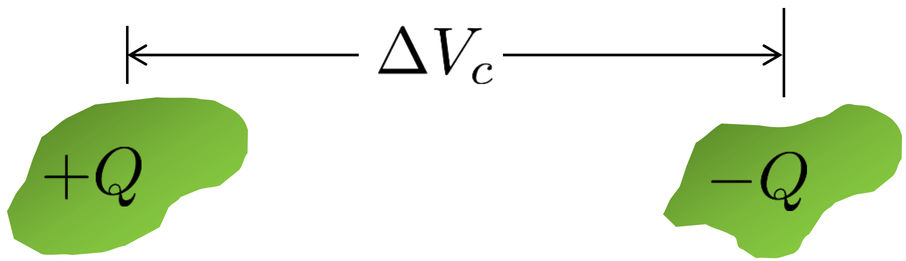
Important facts for a charged conductor in equilibrium

The electric field and equipotentials between two charged conductors



Capacitors and Capacitance

We have already introduced the parallel plate capacitor as a source of uniform electric field, **but in general, a capacitor is any two conductors (electrodes) carrying equal but opposite charges**:



There are complicated electric fields and equipotentials between the conductors that depend on their geometry; however, the potential difference is proportional to the charge:

$$\Delta V_c \propto Q \quad \Rightarrow \Delta V_c = \left(\frac{1}{C}\right) Q$$

So, we define the **Capacitance of the two electrodes as:**

$$C = \frac{Q}{\Delta V_c} \qquad \text{Units: } = \frac{C}{V} \equiv \text{Farad (F)}$$

The **capacitance of a capacitor** is the proportionality constant between the charge on the conductors and the potential between them. It depends on the geometry of the conductors.

PhET Capacitor Demo

Remember from watching the video, **The Story of Electricity Part 1**, that the **Leyden Jar** was a device to store charge - - maybe you've built your own **Leyden jar**. The modern equivalent of the Leyden Jar is the Capacitor. In this sense, the Capacitance of any Capacitor is a measure of how much charge the device can store for a given potential difference, *i.e. it's charge Capacity*.

In this demo, we will explore how the characteristics of a parallel place capacitor affect how much charge it can store, i.e. it's capacitance.

After we explore a bit, we'll develop the necessary physical relations.

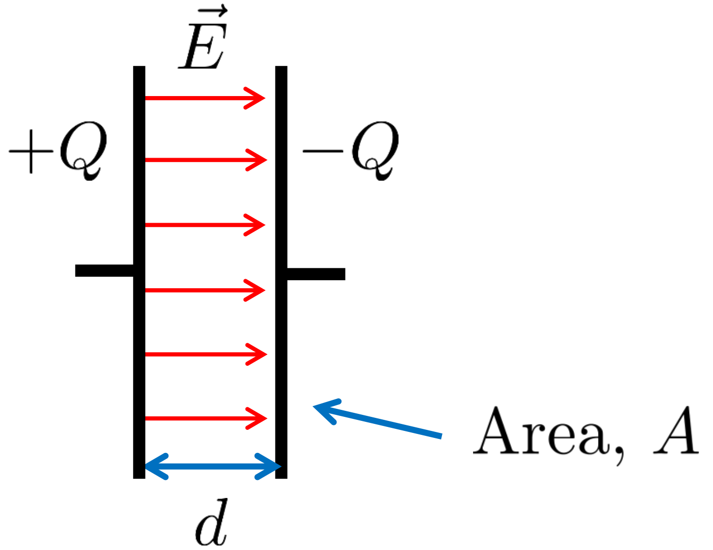
The PhET simulation is at:
https://phet.colorado.edu/en/simulation/capacitor-lab-basics

***By the way**: we'll cover multiple capacitors in combination in chapter 28 (circuits) when it is used.*

The Capacitance of a Parallel Plate Capacitor

In the PhET demo, we found that for a **parallel plate capacitor**, the capacitance is larger for larger plate area and smaller plate separation.

Here, we'll analyze this capacitor geometry:



For a parallel plate capacitor:

Know: $$E = \frac{Q}{\epsilon_0 A} \quad \text{(Uniform)}$$

And for a **<u>uniform</u>** **electric field:**

$$\Delta V = Ed = \left(\frac{Q}{\epsilon_0 A}\right) d$$

So: $$Q = \left(\frac{\epsilon_0 A}{d}\right) \Delta V$$

For a <u>Parallel Plate Capacitor</u>: $$C = \frac{Q}{\Delta V} = \frac{\epsilon_0 A}{d}$$

As you saw in the PhET demo, the plates can hold more charge for larger A and/or smaller d.

Whiteboard Problem: 26-6 If Time

You need to construct a 100 pF capacitor for a science project. You plan to cut two L X L metal squares and insert small spacers between their corners to make a parallel plate capacitor. The thinnest spacers you have are 0.20 mm thick.

What value of L will give you the capacitance you want? (LC)

Energy Stored in a Capacitor

Your author shows that it takes energy to charge a capacitor, and this energy is stored in the capacitor:

Energy Stored in a Capacitor

$$U_c = \frac{Q^2}{2C} = \frac{1}{2}C\left(\Delta V_c^2\right)$$

Where, exactly, does this energy reside? **Consider a parallel plate capacitor:**

$$U_c = \frac{1}{2}C\left(\Delta V_c^2\right) = \frac{1}{2}\left(\frac{\epsilon_0 A}{d}\right)(Ed)^2 = \frac{1}{2}\epsilon_0 A d E^2$$

Now, consider the **Energy Density** - *i.e. the energy per unit volume*:

$$u_E = \frac{U_c}{\text{Volume}} = \frac{\frac{1}{2}\epsilon_0 A d E^2}{Ad} = \frac{1}{2}\epsilon_0 E^2$$

So, the energy is stored in the field with energy density:

$$u_E = \frac{1}{2}\epsilon_0 E^2$$

This is a general result that is true for all electric fields.