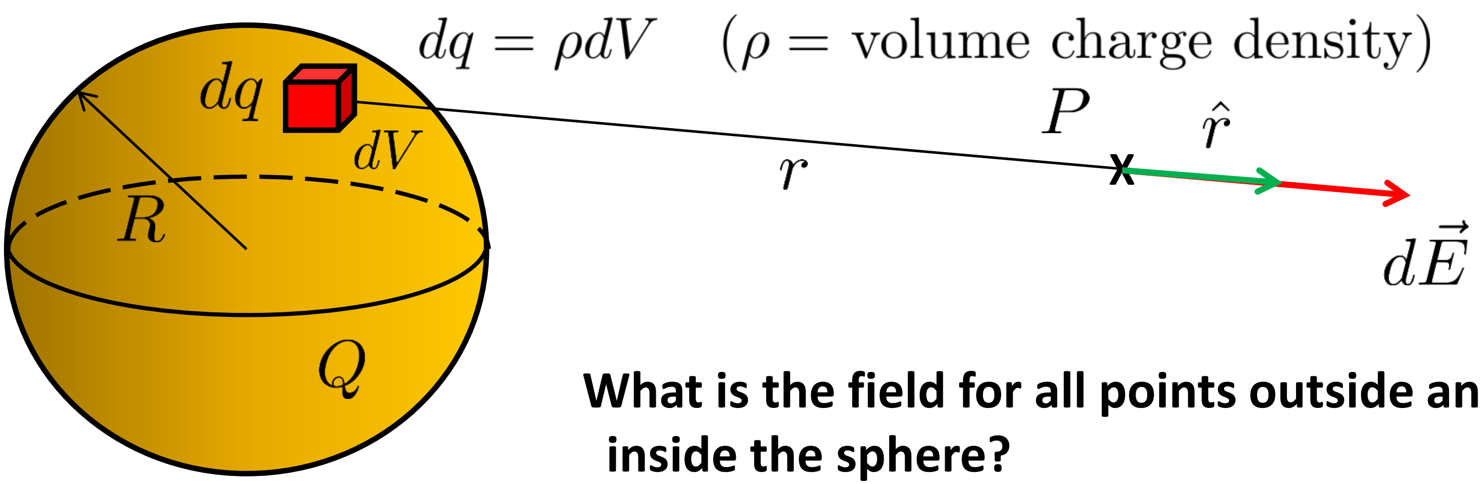
24: Gauss' Law

In this chapter, we return to the problem of finding the electric field for various distributions of charge.

Question: A really important field is that of a uniformly charged sphere, or a charged spherical conductor. **Why haven't we figured out how to do these?**



What is the field for all points outside and inside the sphere?

We could use the point charge integration technique:

$$\text{At P: } \vec{E} = \int_{\text{sphere}} d\vec{E} = \int_{\text{sphere}} \frac{K\,dq}{r^2}\hat{r} = \int_{\text{Vol}} \frac{K\rho dV}{r^2}\hat{r}$$

This works, but it's *pretty darn* difficult!

We'll see that for **highly symmetric bodies**, **Gauss' Law** will give us an easy way to find the electric field at any point in space (inside the body or outside).

Gauss' Law: The Language

Before we even get to Gauss' Law, we have to learn to speak its language. The two important ideas are **symmetry** and **flux**.

Symmetry: In section 24.1, your author covers the necessary aspects of symmetry; read this carefully. The basic idea is that if you do something to an object, and its appearance is unchanged, that's a symmetry transformation of the object.

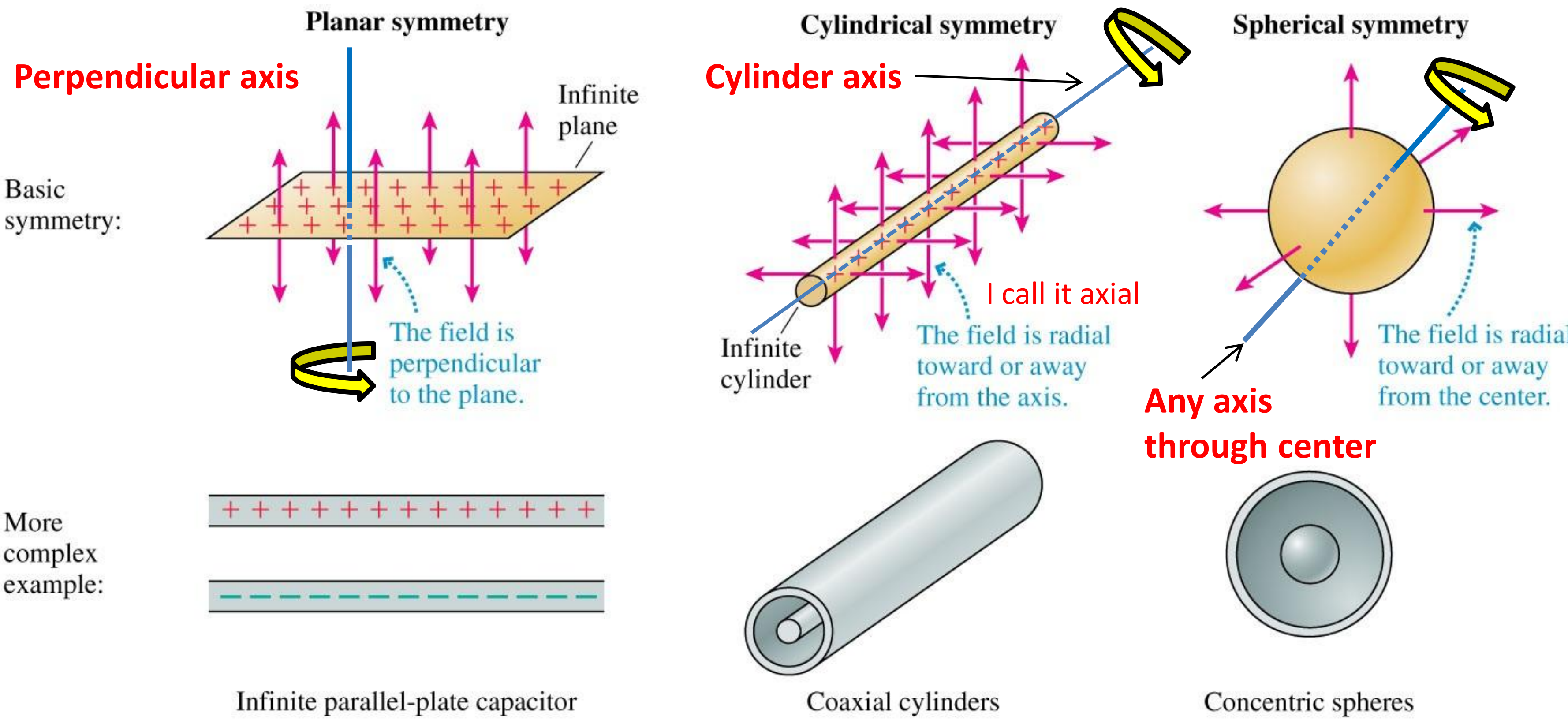
e.g. ***if I rotate a ball about any axis through its center***

or, if I rotate a cylinder about it's axis

or, if I rotate an infinite plane about a perpendicular axis

Important Result: **"The symmetry of the electric field must match the symmetry of the charge distribution."**

Symmetries that we'll use:



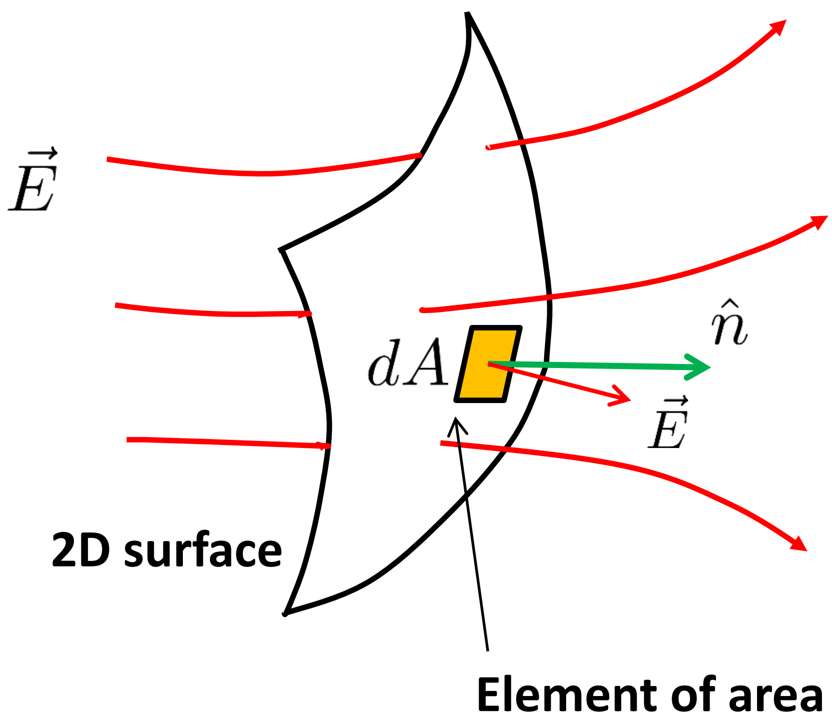
Flux of the Electric Field

Flux is a measure of how much an electric field passes through a 2D surface.

Flux depends on the size and direction of the field and the size, shape, and orientation of the surface; e.g. light rays and shadows:

Your author starts with flux for special cases and builds to a general definition; you should read his account, we'll do the opposite:

Flux for the General Case (any field, any surface):



$\hat{n}$ = unit vector normal to the surface

$$d\vec{A} = \hat{n}\, dA$$

Electric Flux:

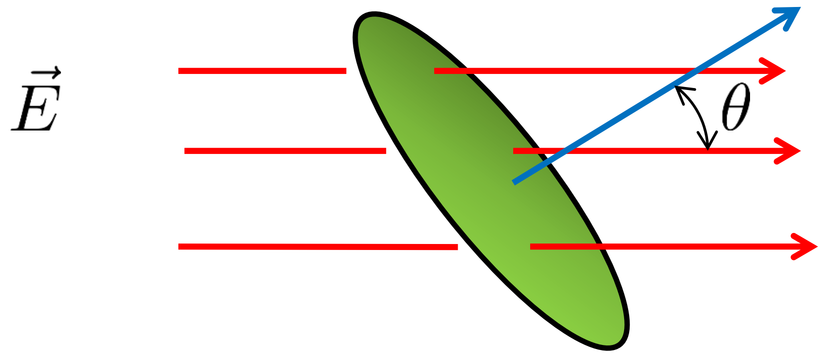
$$\Phi_e \equiv \int_{\text{surface}} \vec{E} \cdot d\vec{A}$$

$$\text{units: } \frac{Nm^2}{C}$$

Note: this is called a **surface integral**

Special Cases of Flux

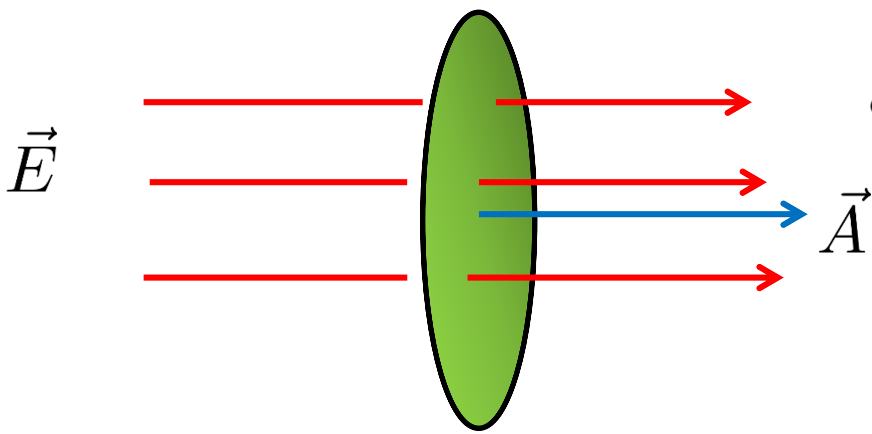
Uniform Electric Field & Planar Surface:



$$\vec{A} = \{A, \perp \text{ to surface}\}$$
$$\text{or, } \vec{A} = A\hat{n}$$

$$\Phi_e = \int \vec{E} \cdot d\vec{A} = \vec{E} \cdot \int d\vec{A} = \vec{E} \cdot \vec{A} = EA\cos\theta$$

Uniform Electric Field & Planar Surface perpendicular to the field:



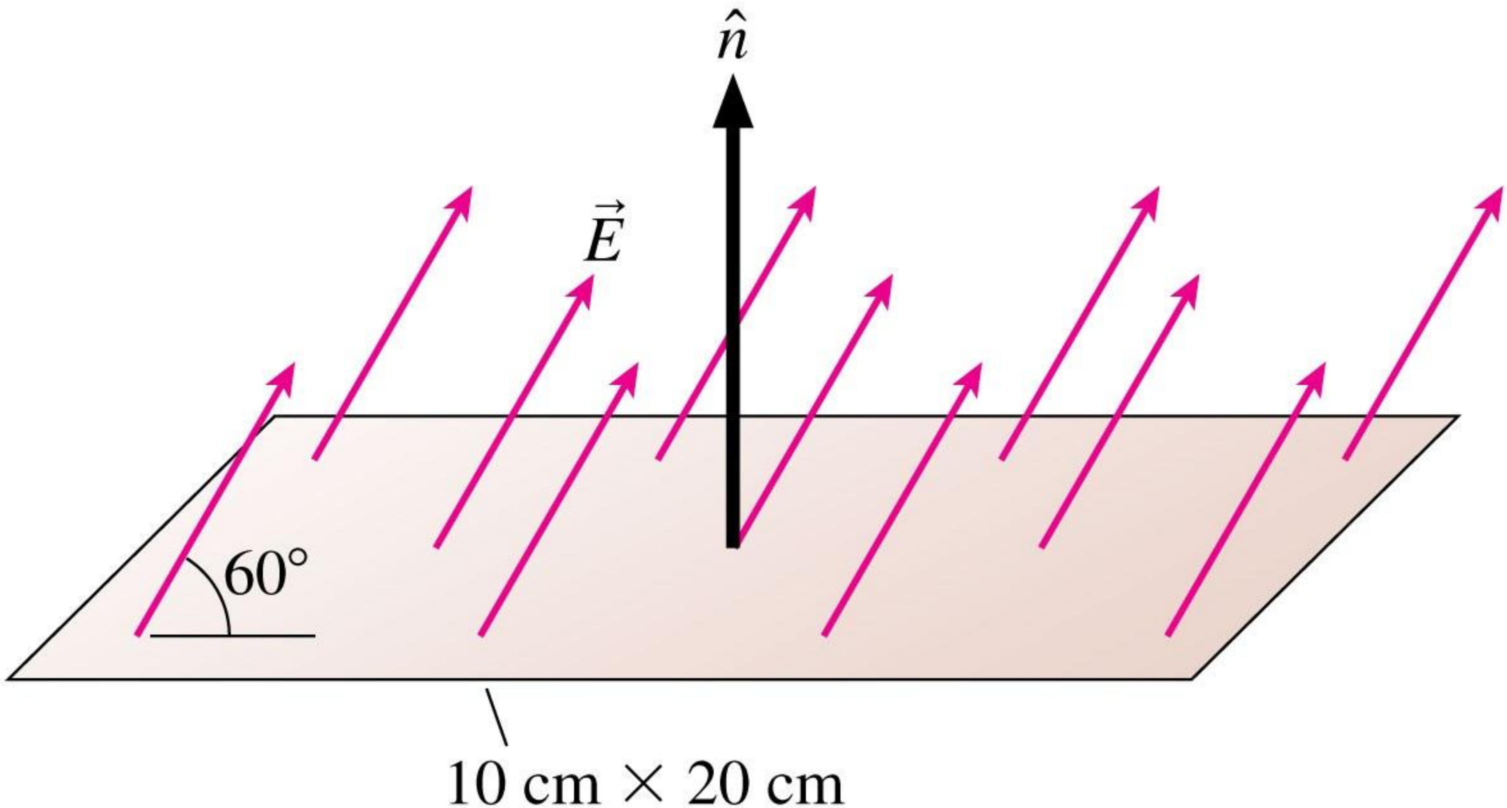
$$\Phi_e = \vec{E} \cdot \vec{A} = EA\cos 0^\circ = EA$$

For either of these cases, the relation to remember for a uniform field and planar area is:

$$\Phi_e = \vec{E} \cdot \vec{A}$$

Whiteboard Problem: 24-1

The electric flux through the surface shown is 25 Nm^2/C.
What is the electric field strength? (LC)



Whiteboard Problem: 24-2

A 2.0 cm X 3.0 cm rectangle lies in the xy-plane. What is the electric flux through the rectangle if:

$$\vec{E} = 100\hat{\imath} + 50\hat{k}\ N/C$$ **(LC)**

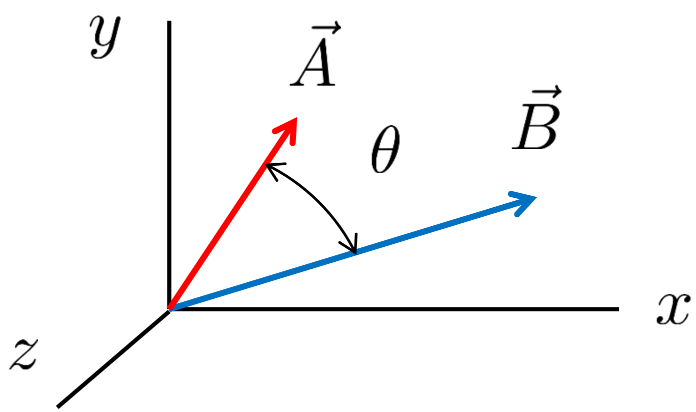
Are you struggling with this?
Having trouble finding the angle between the field and the normal vector? You don't have to.
Maybe we should review the vector dot product?

Quick Review of the Vector Dot Product

For the previous special cases of flux, the equation that you want to remember is for a **uniform field and a planar surface**:

$$\Phi_e = \vec{E} \cdot \vec{A}$$

It is **essential** that you be able to do vector dot (or scalar) products:



When I say "$\vec{A} \cdot \vec{B}$," what pops into your mind?

$$\vec{A} \cdot \vec{B} = AB\cos\theta$$

But, if you know the vectors in **component form for a set of coordinates, then**:

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$$

You have to be able to calculate dot products both ways!

Now, go back and finish WB Problem 24-2.

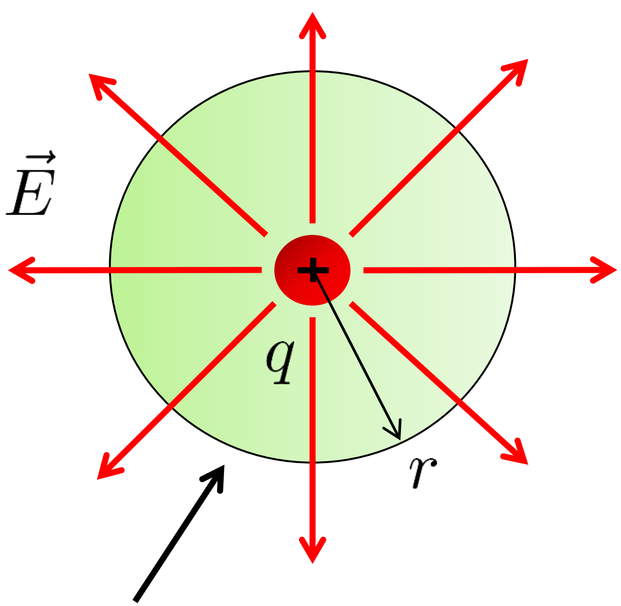
Gauss' Law

Now that we know what the flux of the electric field is and a little about symmetry, we're ready to introduce **Gauss' Law** (Carl Friedrich Gauss).

Gauss noticed a simple and general feature about **all electric fields**:

"The electric field comes out of positive charge, and goes in to negative charge."

And, he noticed for a **point charge field**:



surface of a sphere of radius r with q at the center

Flux through the spherical surface:

$$\Phi_e = \int_{\text{sphere}} \vec{E} \cdot d\vec{A}$$

$$= \int E dA \quad (\vec{E} \perp \text{surface everywhere})$$

$$= E \int dA \quad (E = \text{constant on surface})$$

$$= E 4\pi r^2$$

$$\text{We know: } E = \frac{Kq}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}$$

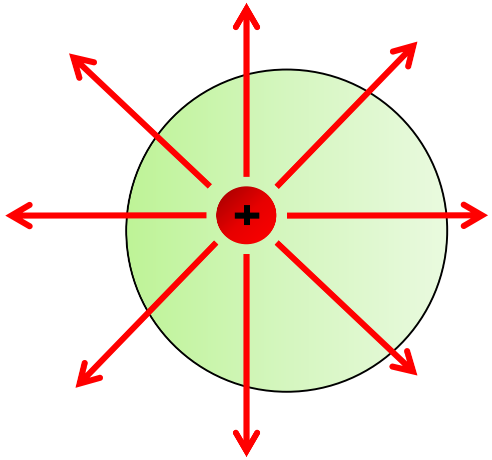
$$\text{So, } \Phi_e = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2} 4\pi r^2 = \frac{q}{\epsilon_0}$$

The flux through the surface depends only on the charge that's inside the surface!

Obvious Questions about Gauss' Law

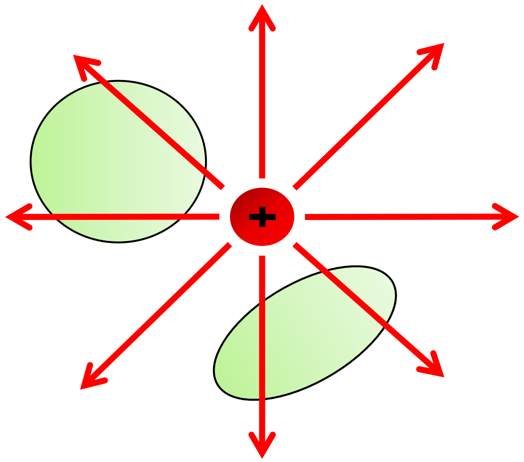
What if the charge isn't at the center of the sphere?

$$\text{Still have: } \Phi_e = \frac{q}{\epsilon_0}$$



What if there is no charge inside the sphere?

With no charge inside, the flux is always zero, as much flux in as out.

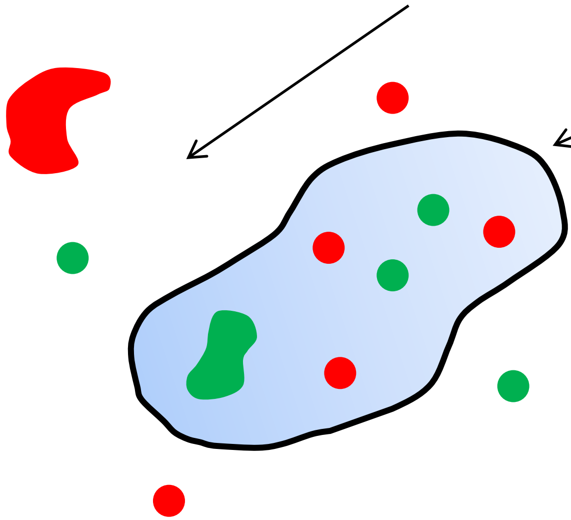


Does the shape of the surface matter?

No, all that matters is the amount of charge that is inside the surface.

Gauss' Law: General Statement

For any distribution of charge and any 2D closed surface S:



Flux through S = $\dfrac{\text{\{Net charge inside S\}}}{\epsilon_0}$

Or:

$$\Phi_e = \oint_S \vec{E} \cdot d\vec{A} = \frac{Q_{\text{in}}}{\epsilon_0}$$

$\oint \Rightarrow$ integral is around the closed surface $\quad Q_{\text{in}} = $ Net charge inside S

(what about the charge outside S?)

Gauss' Law is somewhat odd and abstract – it doesn't just come out and say, "*the field of the charge distribution is this*." Instead, it tells us how the field behaves. It concisely and mathematically:

"Describes how electric charge creates $\vec{E}$"

Gauss' Law for the Electric Field is our First Maxwell Equation.

Whiteboard Problem: 24-3

The figure shows three charges.

Now, on LC, draw the cross sections of two-dimensional closed surfaces through which the electric flux is:

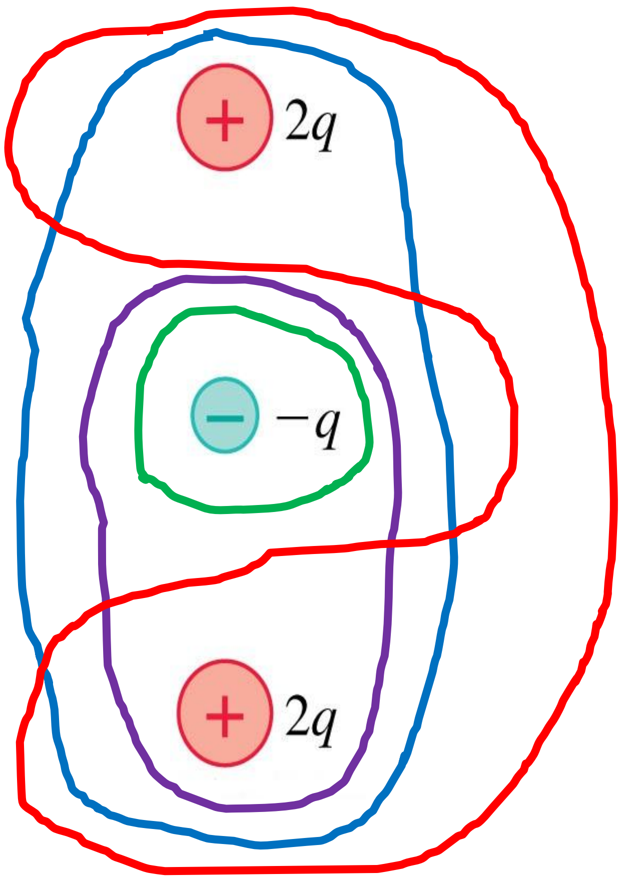
a.) $\frac{-q}{\epsilon_0}$ **(LC)** b.) $\frac{q}{\epsilon_0}$ **(LC)** c.) $\frac{3q}{\epsilon_0}$ **(LC)** d.) $\frac{4q}{\epsilon_0}$ **(LC)**

(a)

(b)

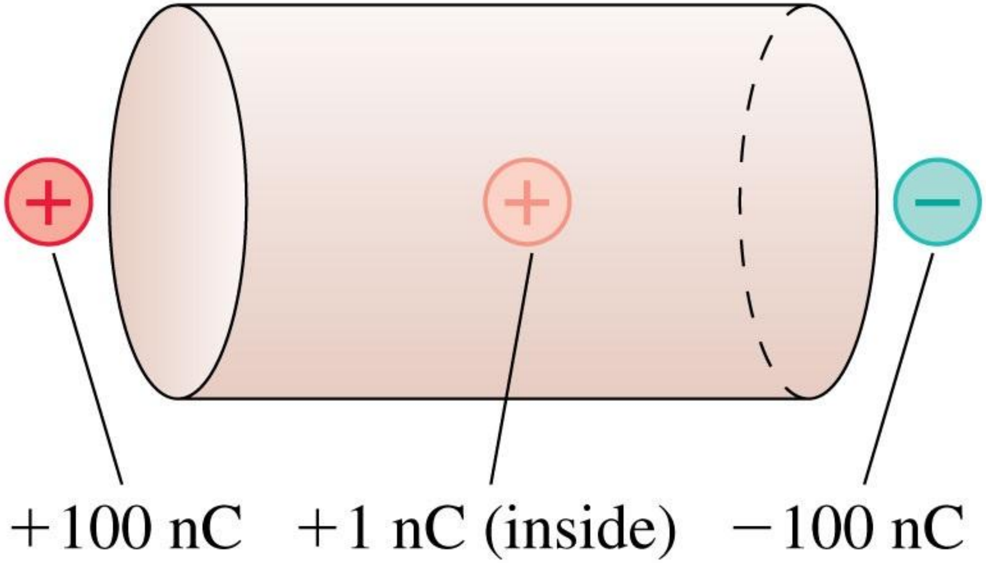
(c)

(d)



Whiteboard Problem: 24-4

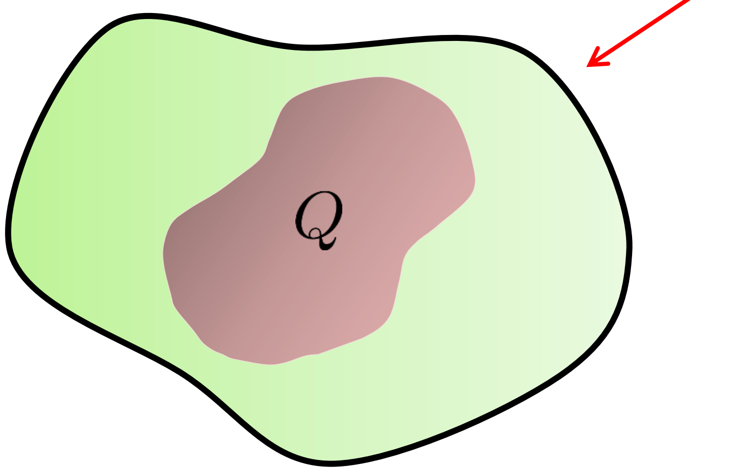
Evaluate $\oint \vec{E} \cdot d\vec{A}$ on the surface of the cylinder. ***(LC)***



I call this the "Meanest Problem in Physics."

You have 30 seconds (but you should only need 10) to complete it!

How do we use Gauss' Law to find the Electric Field?



2D closed **Surface S** that surrounds the charge ("**Gaussian Surface**")

We know: $\Phi_e \text{ (through S)} = \oint_S \vec{E} \cdot d\vec{A} = \frac{Q_{\text{in}}}{\epsilon_0}$ But, how to we get $\vec{E}$?

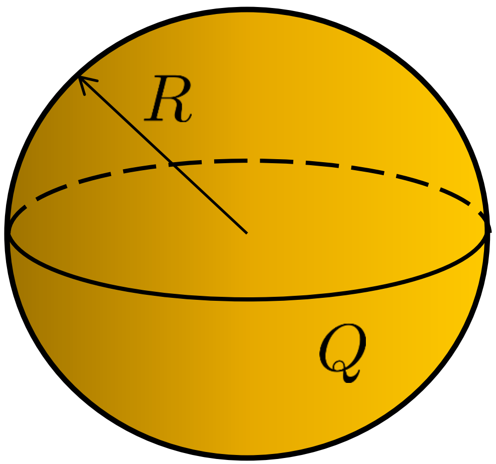
We use the symmetry of the charge to know the symmetry of the field, and we **choose S** so that the integral can be done easily *. . . i.e. in your head!*

Important Point: Gauss' Law is always true for any charge and any surface, but it can only be used to the find the field if the symmetry of the charge allows you to easily determine the flux integral.

Example: A Uniformly Charged Sphere

(Examples 24.3 & 24.4 in the text)

This is the most important problem solved with Gauss' Law:



What is the field for all points outside and inside a uniformly charged sphere of radius R and total charge Q?

$$\text{Uniformly Charged} \Rightarrow \rho = \text{constant}$$

What must the field look like? Use symmetry arguments:

Charge has spherical symmetry ➡ **The field does too.**

$$\text{So, } \vec{E} \text{ is radially out } (Q > 0) \text{ or radially in } (Q < 0)$$

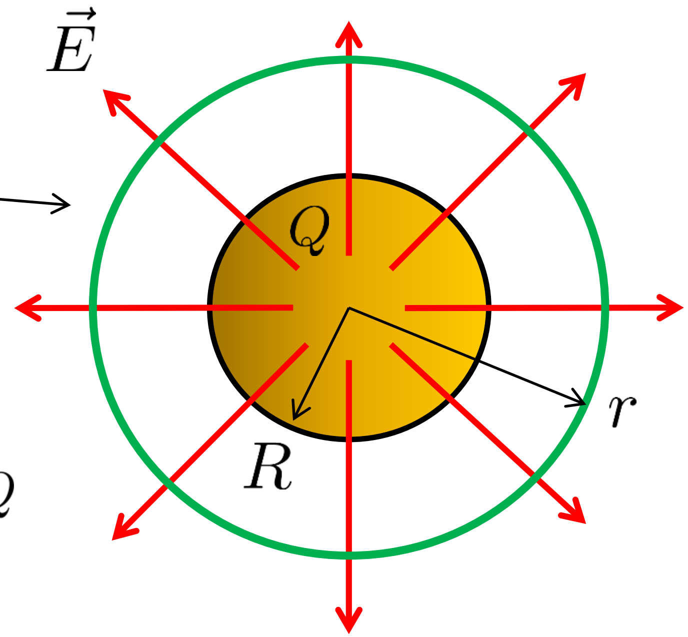
$$\text{also at a given } r,\ |\vec{E}| = \text{constant}$$

Note: we know that the field is radially in or out, but we don't yet know how the magnitude of the field depends on the distance r.

Example: A Uniformly Charged Sphere

Outside the Sphere (r > R):

Choose a **Gaussian Surface** to be a sphere of radius r > R, apply Gauss' Law to it:



$$\Phi_e = \oint \vec{E} \cdot d\vec{A} = \frac{Q_{\text{in}}}{\epsilon_0}$$

work on both sides of the equation

$\vec{E} \parallel d\vec{A}$ everywhere

Charge inside $= Q$

$$\oint E dA = \frac{Q}{\epsilon_0}$$

$|\vec{E}| =$ constant on the surface

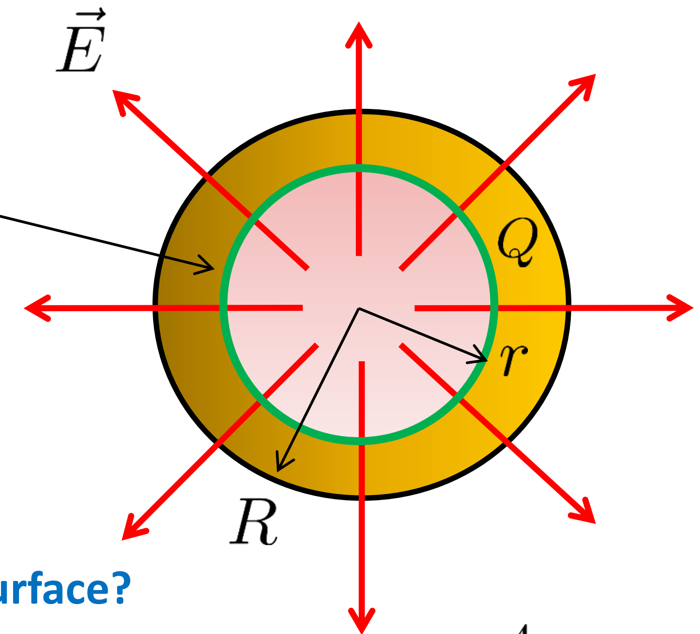
$$E \oint dA = \frac{Q}{\epsilon_0}$$

$$E 4\pi r^2 = \frac{Q}{\epsilon_0}$$

$$\text{So: } E = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \text{ for } r > R$$

The field outside the sphere (r >R) is the same as if all of the charge is concentrated in a point charge at the center. This is true for any spherically symmetric distribution of charge.

Example: A Uniformly Charged Sphere



Inside the Sphere (r < R):

Choose a **Gaussian Surface** to be a sphere of radius r < R, apply Gauss' Law to it:

$$\Phi_e = \oint \vec{E} \cdot d\vec{A} = \frac{Q_{\text{in}}}{\epsilon_0}$$

Using the same arguments as above for r > R,

$$\oint \vec{E} \cdot d\vec{A} = EA = E4\pi r^2$$

But how much charge is enclosed by the Gaussian surface?

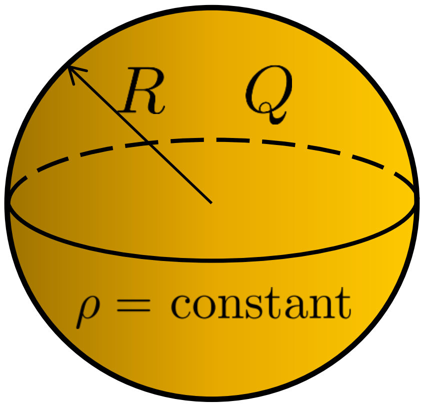
$$Q_{\text{in}} = \rho \text{ (Volume of charge enclosed by Gaussian surface)} = \rho \frac{4}{3}\pi r^3$$

$$\text{Now, } \rho = \frac{\text{Total Charge}}{\text{Total Volume}} = \frac{Q}{\frac{4}{3}\pi R^3} \Rightarrow Q_{\text{in}} = \left(\frac{Q}{\frac{4}{3}\pi R^3}\right)\left(\frac{4}{3}\pi r^3\right) = Q\frac{r^3}{R^3}$$

Putting it together in Gauss' Law: $E4\pi r^2 = \frac{Q_{\text{in}}}{\epsilon_0} = \frac{Q}{\epsilon_0}\frac{r^3}{R^3}$

$$E = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^3} r \text{ for } r < R$$

Example: A Uniformly Charged Sphere

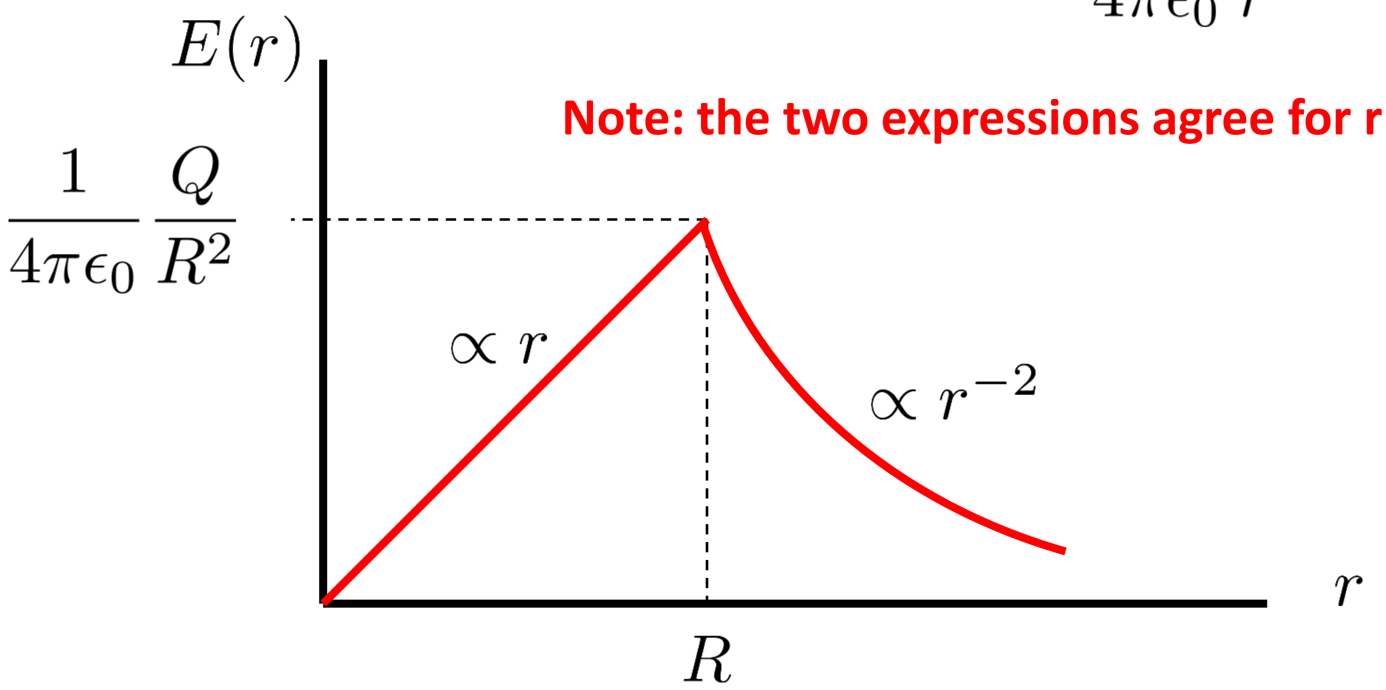


Results:

$$\text{for } r \leq R: \quad E = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^3}r$$

$$\text{for } r \geq R: \quad E = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}$$

Note: the two expressions agree for r = R



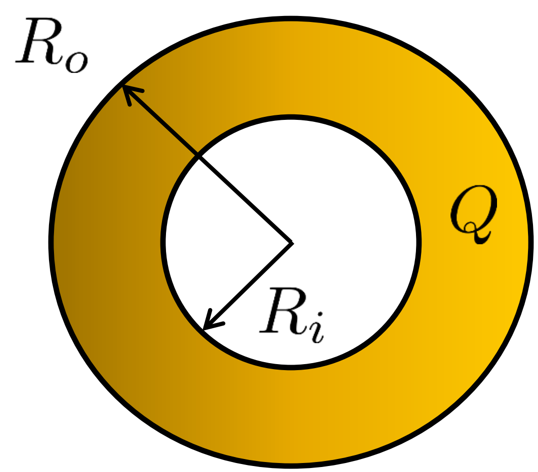
Whiteboard Problem: 24-6

A spherical shell has inner radius R_i and outer radius R_o. The shell contains total charge Q, uniformly distributed. The interior of the shell is empty of charge and matter.

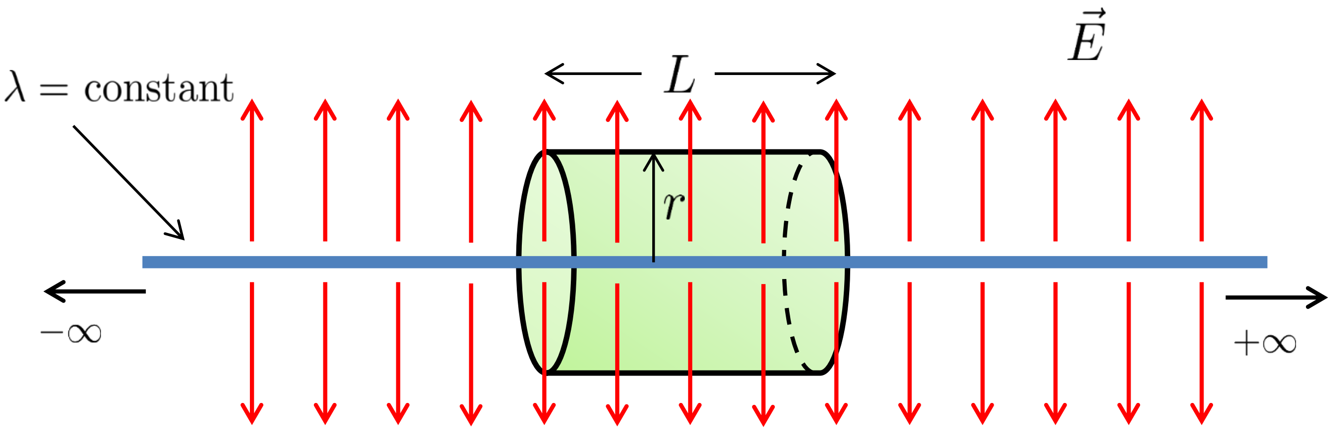
a) **Find the electric field outside the shell, $r > R_o$. (LC)**
b) **Find the electric field in the interior of the shell, $r < R_i$. (LC)**
c) **Find the electric field within the shell, $R_i < r < R_o$. (LC, 2 pts.)**
d) **Do your expressions for the field match at $r = R_i$ and $r = R_o$? (*They must*)**

(a & b are pretty easy, c is tough, follow the same steps that we used for the uniform sphere; *use the handout as a guide*.)

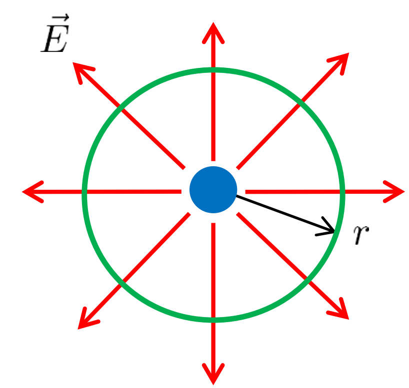
Hint: Draw ,draw . . . lots of pictures!



Cylindrical Geometry e.g: the infinite line of charge



Side View



For cylindrical symmetry, the field must be everywhere away from the axis (positive charge) or everywhere toward the axis (negative charge).

Choose as a Gaussian surface a cylinder of length L and radius r, and apply Gauss' Law:

$\vec{E} \perp$ curved surface and $\parallel$ to ends

$$\Phi_e = \oint \vec{E} \cdot d\vec{A} = \frac{Q_{\text{in}}}{\epsilon_0}$$

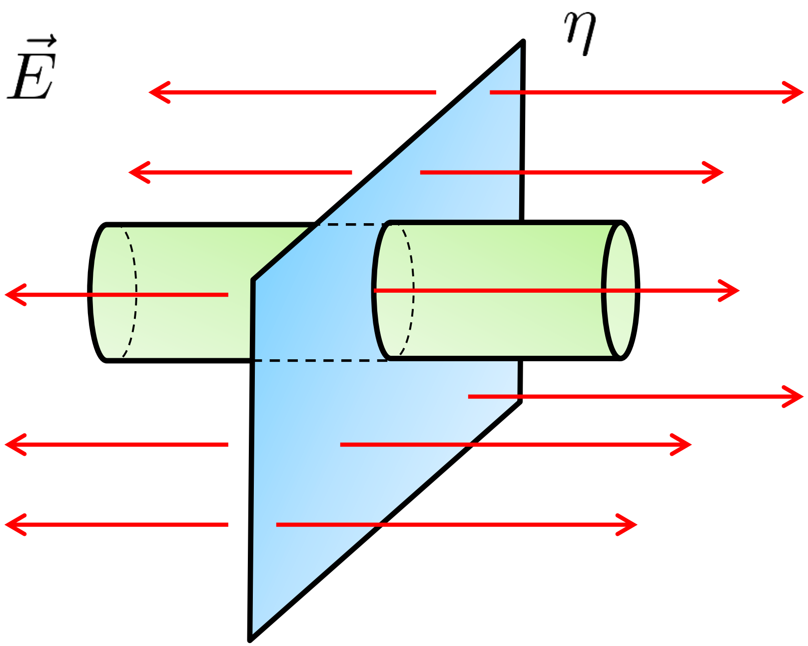
$$Q_{\text{in}} = \text{part of line inside cylinder} = L|\lambda|$$

$$\Phi_e \text{ (left end)} + \Phi_e \text{ (right end)} + \Phi_e \text{ (curved surface)} = 2\pi r L E = \frac{L|\lambda|}{\epsilon_0}$$

So: $$E = \frac{|\lambda|}{2\pi\epsilon_0 r} = \frac{2K|\lambda|}{r}$$

Same thing we got in chapter 23.

Planar Geometry e.g. Infinite Sheet of Charge



For this symmetry, the field must be uniform and point away from the sheet for positive charge and toward the sheet for negative charge.

Choose a Gaussian surface to be a cylinder with **cross sectional area A** and axis parallel to the field. **Apply Gauss' Law:**

$\vec{E} \parallel$ curved surface and $\perp$ to ends

$$\Phi_e = \oint \vec{E} \cdot d\vec{A} = \frac{Q_{\text{in}}}{\epsilon_0}$$

$$Q_{\text{in}} = \text{part of surface inside cylinder} = |\eta| A$$

$$\Phi_e \text{ (left end) } + \Phi_e \text{ (right end) } + \Phi_e \text{ (curved surface) } = EA + EA = \frac{|\eta| A}{\epsilon_0}$$

So: $$E = \frac{|\eta|}{2\epsilon_0}$$

Same thing we got in chapter 23.