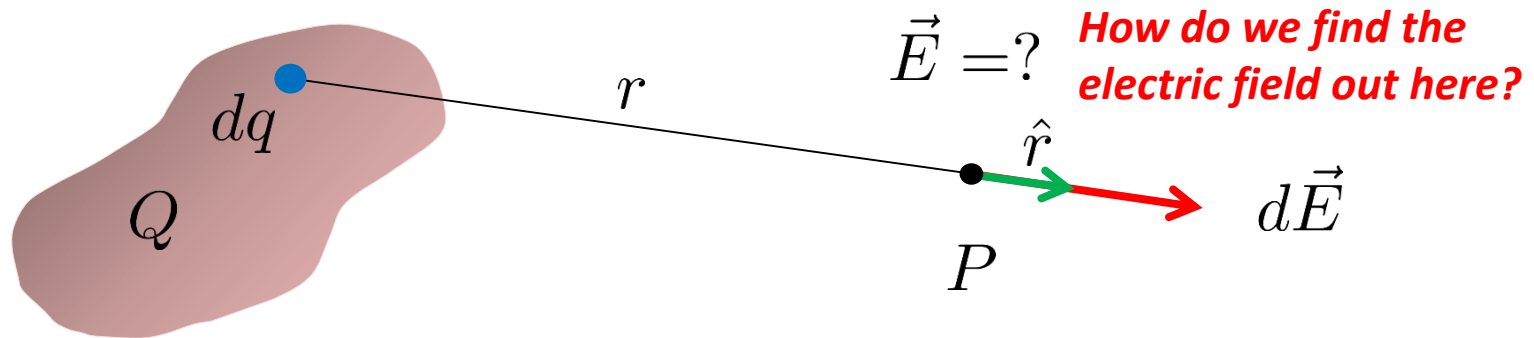


23-2: Electric Field for a Continuous Distribution of Charge

So far, all of the fields that we have been able to determine are for point charges only. **What if we have a real object (like the pie pan or styrofoam slab that you'll play with in lab) that is charged; how would we find it's electric field at some point in space?**

For example, suppose that we have “*charged a baked potato*” to total charge Q :



We treat the charge Q as a collection of infinitesimally tiny charges, dq , (*that's a differential, i.e. an infinitesimally small piece of charge - it's not d times q*) that can be treated as point charges. Each dq creates a infinitesimal point charge field at P :

$$d\vec{E} = K \frac{dq}{r^2} \hat{r}$$

Then we add up all of the $d\vec{E}$'s created by all of the dq 's in the total charge Q .

For an infinite number of infinitesimal small charge elements, this addition becomes an integration:

$$\vec{E}(\text{at } P) = \int d\vec{E} = \int_{(\text{over } Q)} \frac{K dq}{r^2} \hat{r} \quad (\text{"Point Charge Integration"})$$

Electric Field for a Continuous Distribution of Charge

These are the most difficult problems that we'll do in PHY182;

... **so let me ask you a question:**



What did they say?

- Aragorn: *"Are you frightened?"*
- Frodo: *"Yes"*
- Aragorn: *"Not nearly frightened enough. I know what hunts you!"*

Let's watch it again.

I know what hunts you:

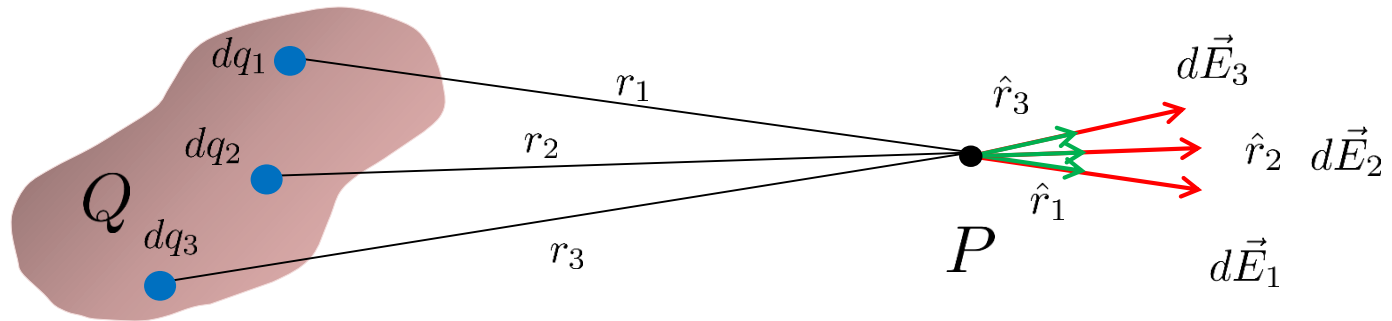
Terrifyingly difficult integrals in black cloaks with daggers!

LC: In one word, describe your level of fear. [Here's a similar exchange from Star Wars.](#)

Electric Field for a Continuous Distribution of Charge

The Difficult Part: . . . *it's not the integral, but getting to the correct integral!*

For example: Consider three elements of charge:



Each dq has a different r and \hat{r} .

So, before we can integrate:
$$\vec{E} = \int \frac{K dq}{r^2} \hat{r}$$

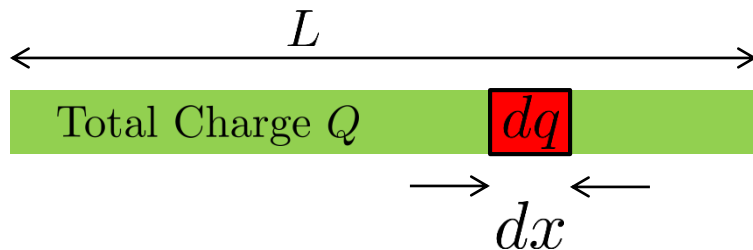
we have to express dq in terms of Q and r and \hat{r} in terms of the coordinates of each dq and P .

- this is the difficult part!

Elements of Charge, dq in terms of Q

For objects with a **uniform distribution of charge**, we can express the element of charge, dq , in terms of the total charge, Q , using a **charge density**.

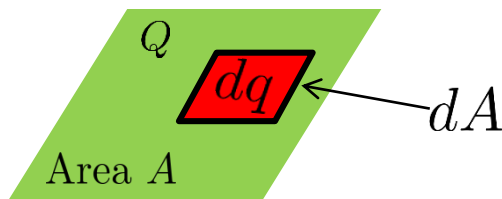
One Dimensional Linear Charge Density:



linear charge density, $\lambda = \frac{Q}{L} = \text{constant}$
[units = C/m]

$$dq = \lambda dx$$

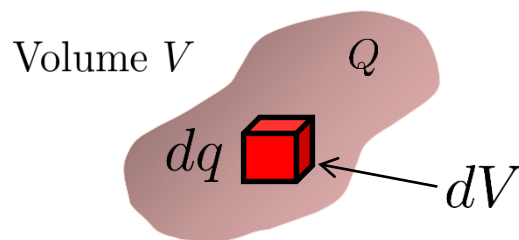
Two Dimensional Surface Charge Density:



surface charge density, $\eta = \frac{Q}{A} = \text{constant}$
[units = C/m²]

$$dq = \eta dA$$

Three Dimensional Volume Charge Density:



volume charge density, $\rho = \frac{Q}{V} = \text{constant}$
[units = C/m³]

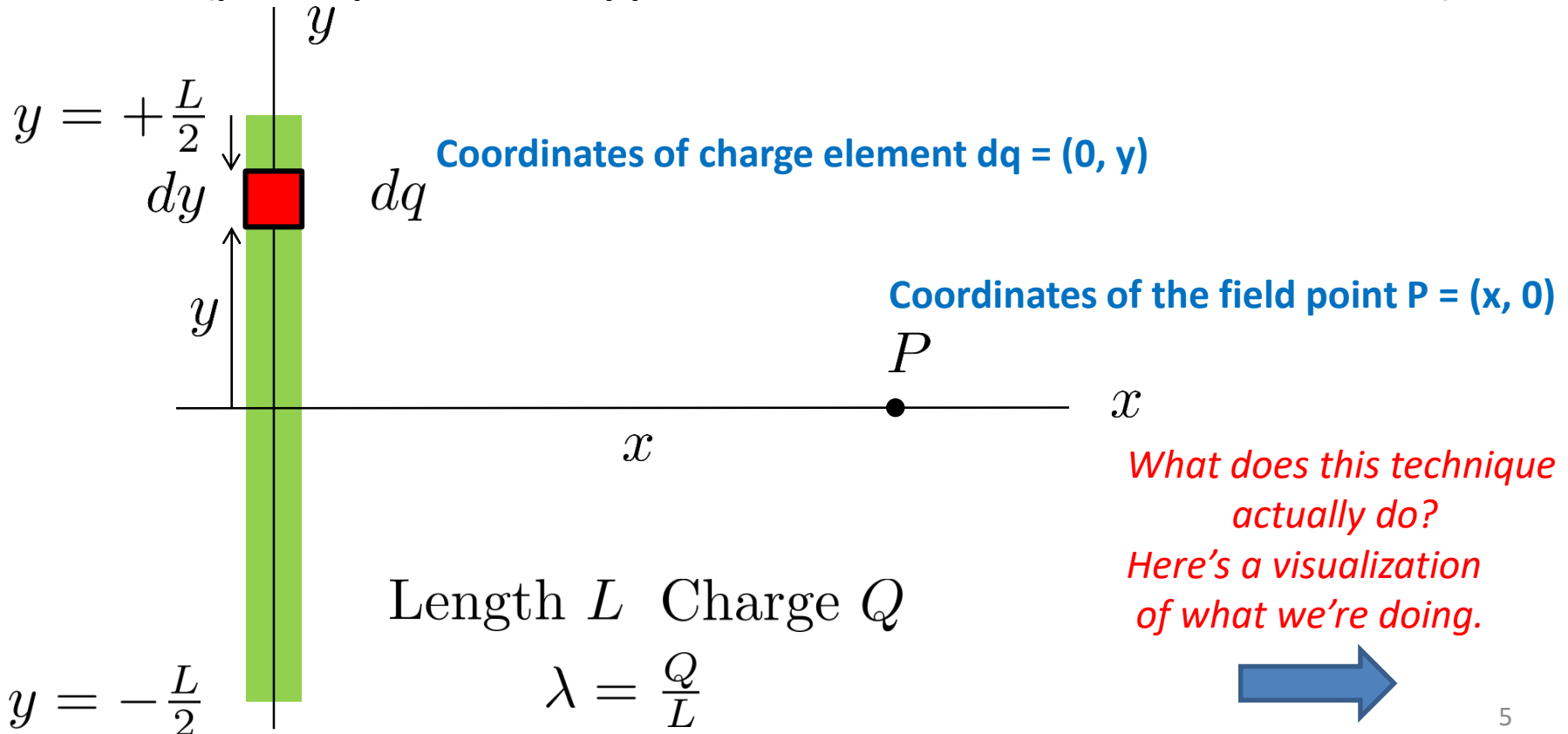
$$dq = \rho dV$$

The Electric Field of a Rod on the Bisecting Axis

This is example 23-3 in your text. We'll work it out together a little differently than your author does. Then you'll try a similar one as a whiteboard problem.

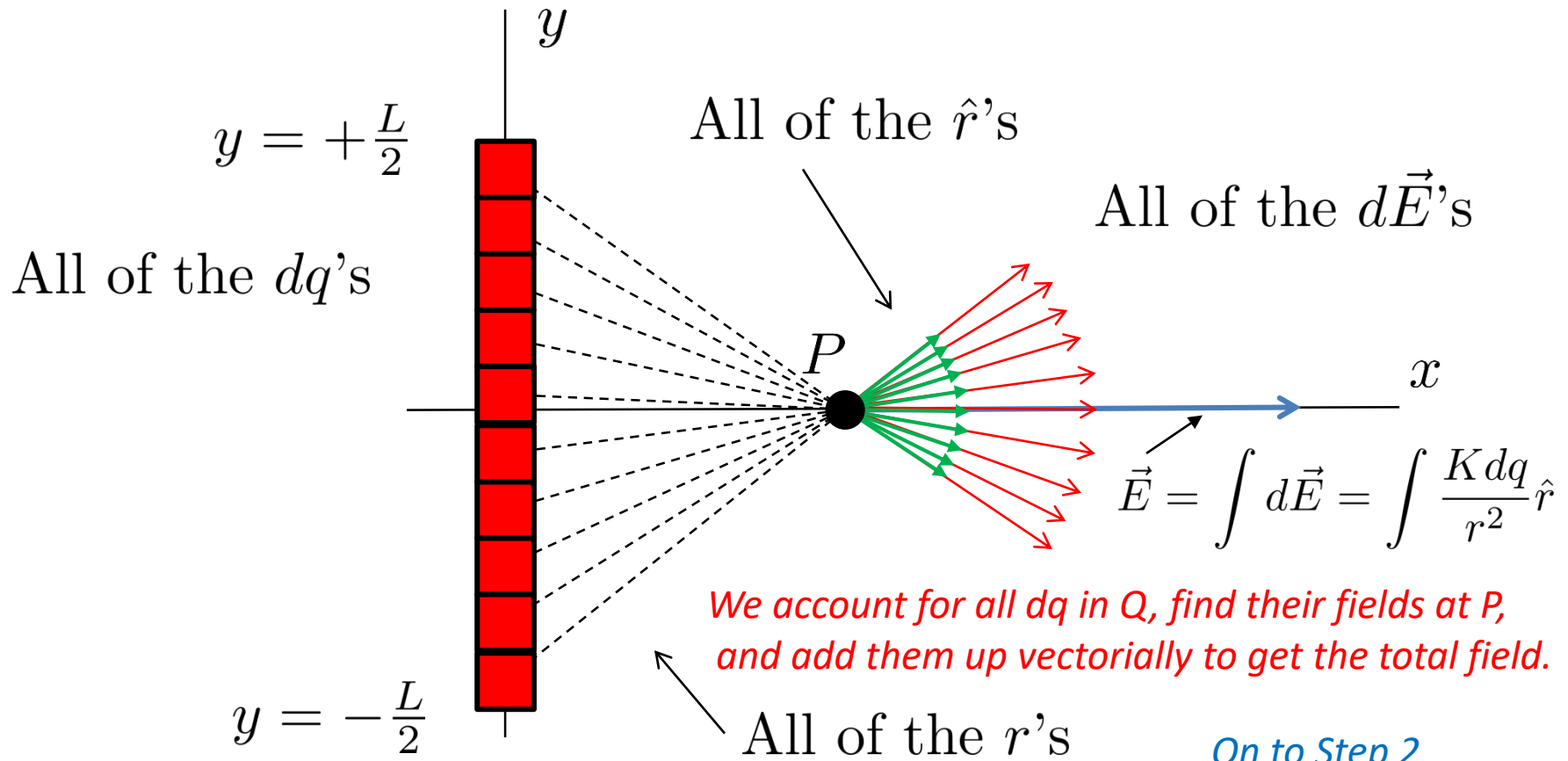
Here, we have a uniformly charged rod of length L and charge Q , and we want to find the electric field at P on the perpendicular axis through the center of the rod.

Step 1: Draw it, choose coordinates, and select a charge element dq (place dq at an arbitrary point on the rod, but not at the ends or middle):



A Visualization of What Point Charge Integration Does

The point charge integration technique adds up all of the fields created by an infinite number of charge elements dq . This slide shows a visualization of what the integration means for a finite number of dq comprising a rod of charge.

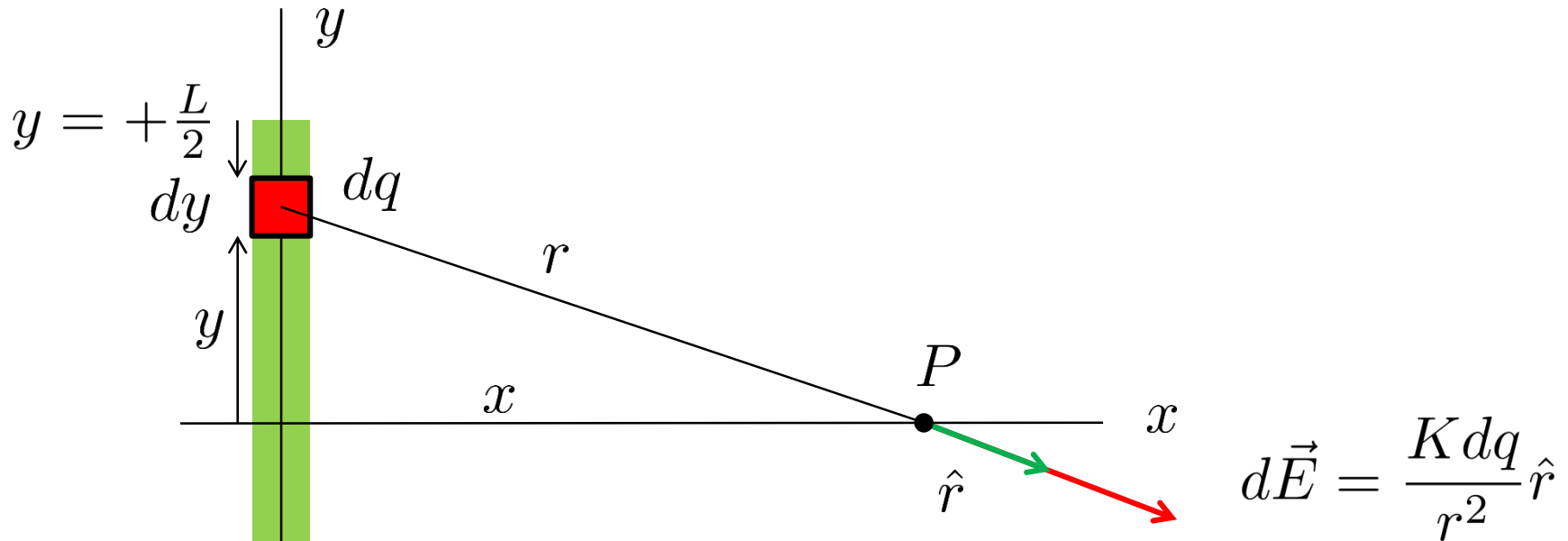


Of course, in the actual problem, there are an infinite number of infinitesimally small $d\vec{E}$'s – that's integration.



The Electric Field of a Rod on the Bisecting Axis

Step 2: write the field created by the charge element (treat as a point charge)
at the field point P.

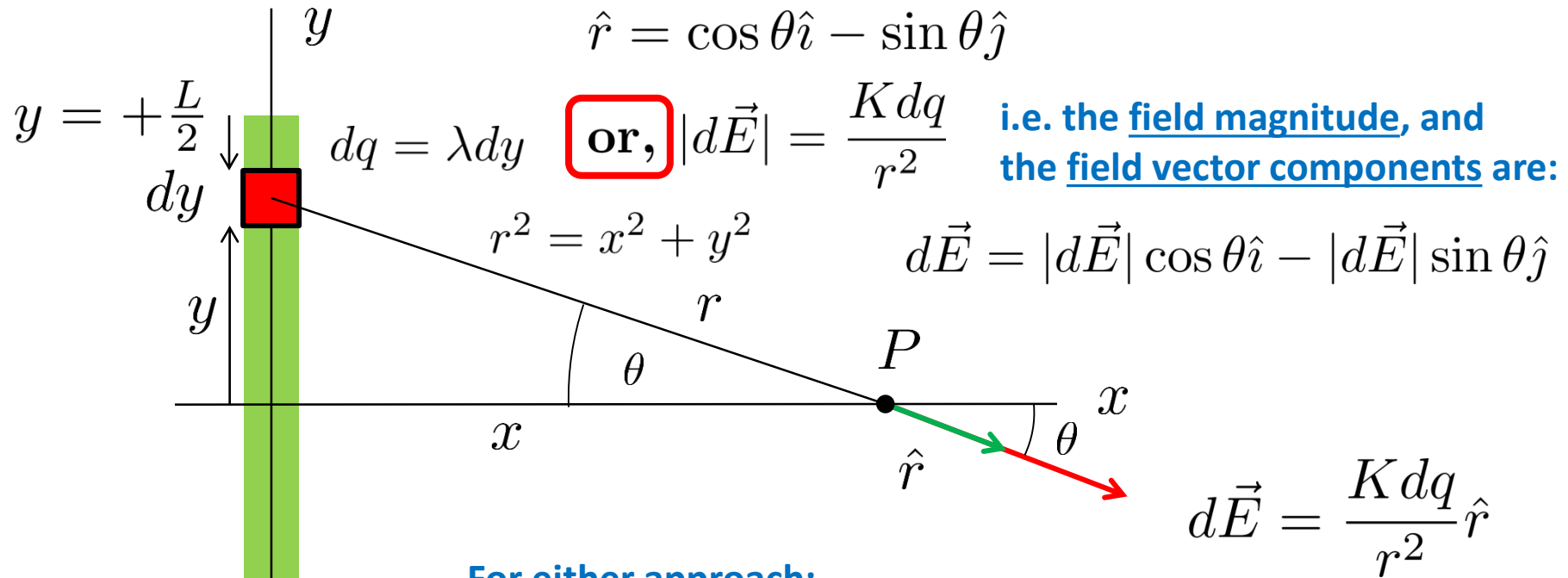


Now, to integrate $d\vec{E}$, we have to express dq in terms of Q , and r and \hat{r} in terms of the coordinates of P and dq .

$$dq = \lambda dy \quad r^2 = x^2 + y^2$$

The Electric Field of a Rod on the Bisecting Axis

Step 3: write \hat{r} in terms of x and y or write out the components of $d\vec{E}$



For either approach:

$$d\vec{E} = \frac{K dq}{r^2} \cos \theta \hat{i} - \frac{K dq}{r^2} \sin \theta \hat{j}$$

Now from geometry:

$$\cos \theta = \frac{x}{r} = \frac{x}{\sqrt{x^2 + y^2}} \qquad \sin \theta = \frac{y}{r} = \frac{y}{\sqrt{x^2 + y^2}}$$

The Electric Field of a Rod on the Bisecting Axis

Step 4: Bring it all together:

On the previous slide, we had:
$$d\vec{E} = \frac{Kdq}{r^2} \cos \theta \hat{i} - \frac{Kdq}{r^2} \sin \theta \hat{j}$$

Make the substitutions:
$$d\vec{E} = \frac{K\lambda dy}{(x^2 + y^2)} \frac{x}{\sqrt{x^2 + y^2}} \hat{i} - \frac{K\lambda dy}{(x^2 + y^2)} \frac{y}{\sqrt{x^2 + y^2}} \hat{j}$$

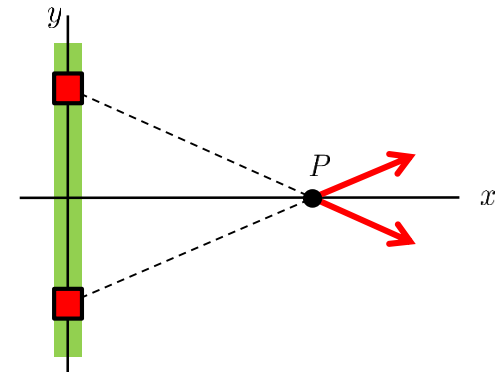
Simplify:
$$= \frac{K\lambda x dy}{(x^2 + y^2)^{\frac{3}{2}}} \hat{i} - \frac{K\lambda y dy}{(x^2 + y^2)^{\frac{3}{2}}} \hat{j} = dE_x \hat{i} - dE_y \hat{j}$$

Name the components:

Step 5: Now, we want to add up the fields produced by all of the dq's. That means that we'll integrate each component separately over the length of the rod from $y = -L/2$ to $y = +L/2$.

y – component:

Your author uses a **symmetry argument*** to show that the y-component integrates to zero. For every charge element on the positive y-axis, there's another one on the negative y-axis that has an oppositely directed field y-component. These two y-components will add to zero.



Or, you could just integrate it to find:
$$E_y = \int dE_y = \int_{-L/2}^{+L/2} \frac{K\lambda y dy}{(x^2 + y^2)^{\frac{3}{2}}} = 0$$

* Be careful with symmetry arguments; not all problems have a symmetry.

The Electric Field of a Rod on the Bisecting Axis

Step 5. (continued) x – component integral:

$$E_x = \int dE_x = \int_{-L/2}^{+L/2} \frac{K \lambda x dy}{(x^2 + y^2)^{\frac{3}{2}}} = K \lambda x \int_{-L/2}^{+L/2} \frac{dy}{(x^2 + y^2)^{\frac{3}{2}}}$$

How do we do this integral? Use the [Integral Table in Appendix A*](#)

In the table we find: $\int \frac{dx}{(x^2 \pm a^2)^{\frac{3}{2}}} = \frac{\pm x}{a^2 \sqrt{x^2 \pm a^2}} \quad a = \text{constant}$

Be careful translating from the table to our integral:

<u>table</u>	<u>our integral</u>
$x \longleftrightarrow y$	
$a \longleftrightarrow x$	

So, our integral is:

$$E_x = K \lambda x \left[\frac{y}{x^2 \sqrt{x^2 + y^2}} \right]_{y=-L/2}^{y=+L/2}$$

**Yes, you will be given a copy of this table for the exam!*

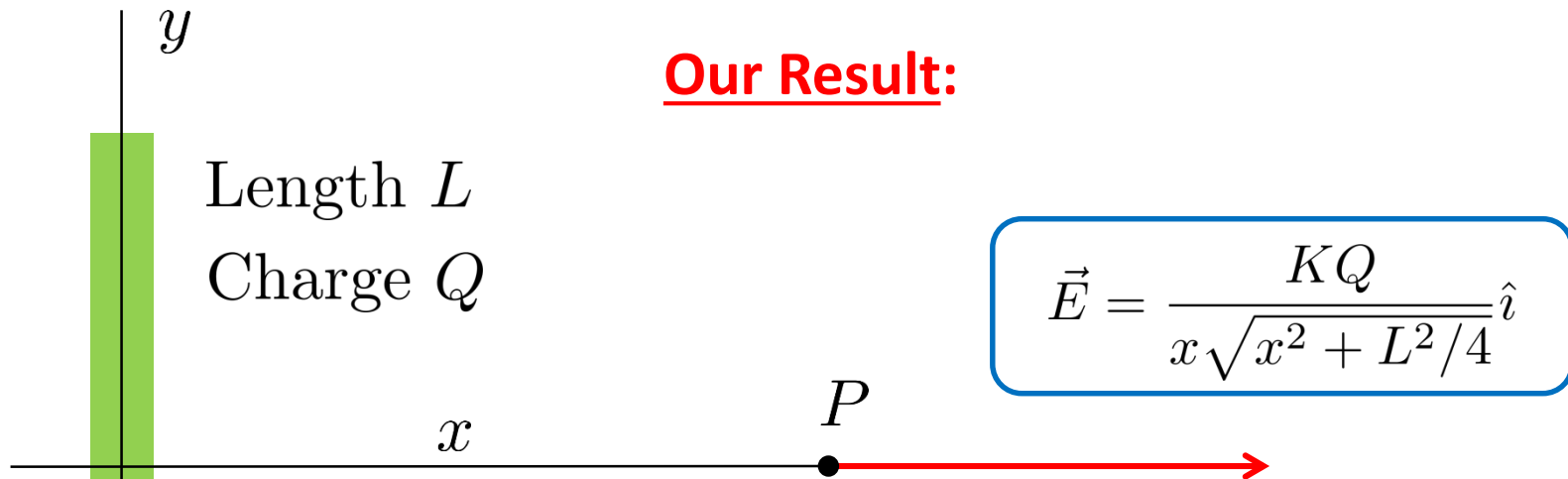
The Electric Field of a Rod on the Bisecting Axis

Step 6: Do the arithmetic

$$\begin{aligned} E_x &= K\lambda x \left[\frac{y}{x^2 \sqrt{x^2 + y^2}} \right]_{y=-L/2}^{y=+L/2} \\ &= \frac{K\lambda}{x} \left[\frac{L/2}{\sqrt{x^2 + L^2/4}} - \frac{-L/2}{\sqrt{x^2 + L^2/4}} \right] \\ &= \frac{K\lambda L}{x\sqrt{x^2 + L^2/4}} \left[\frac{1}{2} - \left(-\frac{1}{2} \right) \right] \\ &= \frac{K\lambda L}{x\sqrt{x^2 + L^2/4}} \end{aligned} \quad Q = \lambda L$$

$$\text{So, } E_x = \frac{KQ}{x\sqrt{x^2 + L^2/4}}$$

The Electric Field of a Rod on the Bisecting Axis



Step 7: A very important point: how does the field behave very far from the charge?

For $x \gg L \Rightarrow \vec{E} = \frac{KQ}{x\sqrt{x^2 + L^2/4}} \hat{i} \rightarrow \frac{KQ}{x^2} \hat{i}$

ignore since $x \gg L$

**What is this?
The field of
a point charge!**

“Any charge distribution viewed from a distance \gg size of the charge, looks like a point charge.”

LC: In one word, describe what you think of the solution for this problem.

Whiteboard Problem 23-3

We have just found the electric field vector of a uniformly charged rod of length L and total charge Q on an axis that bisects its center.

Now, you are to find the electric field vector on an axis perpendicular to the rod but passing through its end.

Give your answer in component form. (Enter your expression in LC;

Note: this is a three point shot)

Follow all of the steps that we did in the example:

Step 1: Draw it, choose coordinates, and select a dq

**Step 2: write the field created by the charge element
(treat as a point charge) at the field point**

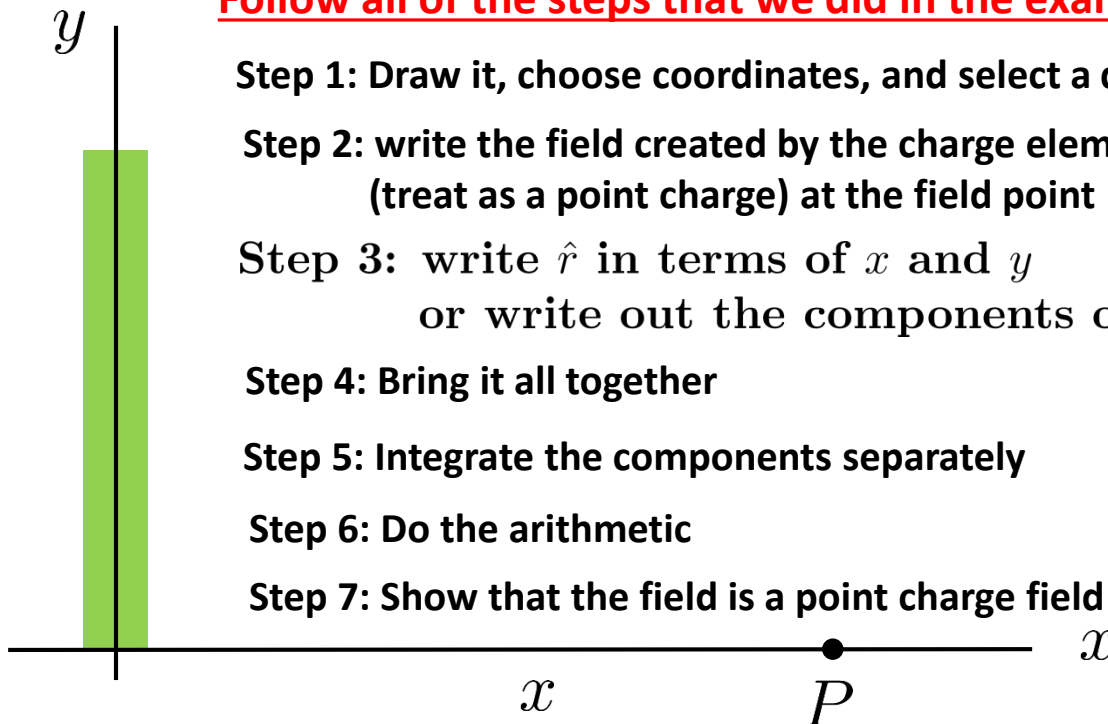
**Step 3: write \hat{r} in terms of x and y
or write out the components of $d\vec{E}$**

Step 4: Bring it all together

Step 5: Integrate the components separately

Step 6: Do the arithmetic

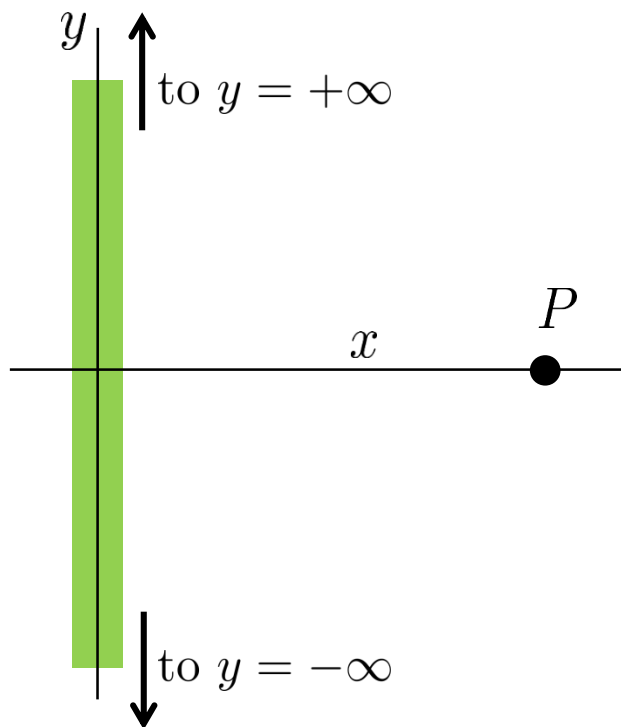
Step 7: Show that the field is a point charge field for $x \gg L$.



Important Field Results

Your author gives the results for many important fields. We won't derive all of them here, but we will use them in problems. ***In several cases, the derivations are very easy using Gauss' Law in Chapter 24.***

The Infinite Line of Charge:



Here, we can use what we had for the finite rod on the bisecting axis, and just change the limits on the integral for the x-component of the field:

$$E_x = K\lambda x \int_{-\infty}^{+\infty} \frac{dy}{(x^2 + y^2)^{\frac{3}{2}}}$$

$$E_x = \frac{K\lambda}{x} \left[\frac{y}{\sqrt{x^2 + y^2}} \right]_{y=-\infty}^{y=+\infty}$$

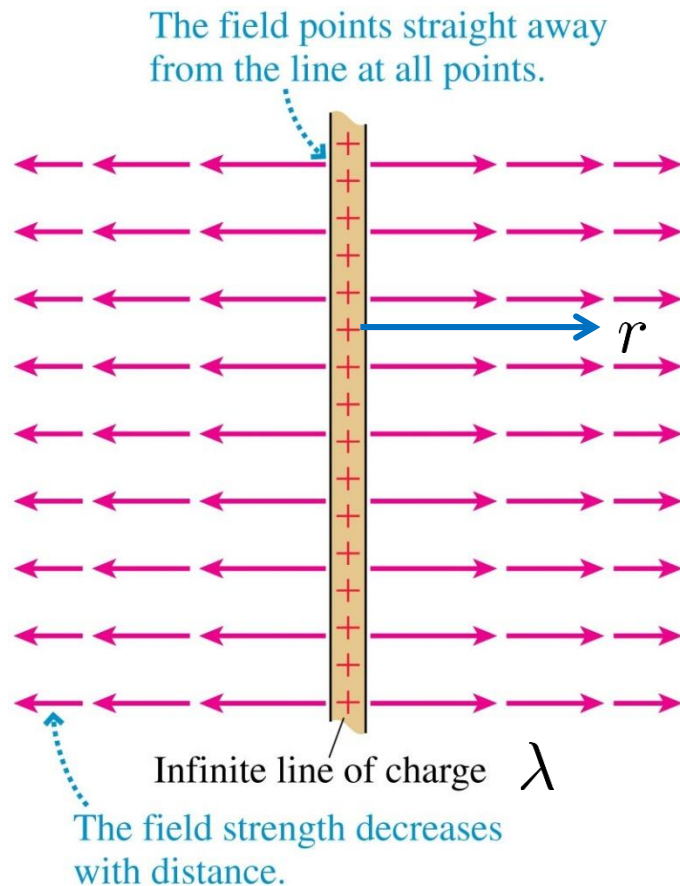
$$\lim_{y \rightarrow \infty} \frac{y}{\sqrt{x^2 + y^2}} = 1$$

$$\lim_{y \rightarrow -\infty} \frac{y}{\sqrt{x^2 + y^2}} = -1$$

$$\text{So, } E_x = \frac{K\lambda}{x} (1 - (-1)) = \frac{2K\lambda}{x}$$

Important Field Results

The Infinite Line of Charge, Summary:



$$\vec{E} = \left\{ \begin{array}{l} \frac{2K|\lambda|}{r}, \text{ away for } \lambda > 0 \\ \frac{2K|\lambda|}{r}, \text{ toward for } \lambda < 0 \end{array} \right\}$$

Note: this is different than a point charge field:

$$\vec{E}_{\text{point charge}} \propto \frac{1}{r^2}; \quad \vec{E}_{\text{inf line charge}} \propto \frac{1}{r}$$

Also, remember that in 3D, the field lines look like the bristles on a cylindrical brush.



Where would we get an infinite line of charge?

A rod of finite length can be considered infinite if r is small compared to the rod length, and the point is not near either end of the rod.

Whiteboard Problem 23-4

The figure below shows two equal but oppositely charged infinite lines of charge that extend out of the screen.

Find an expression for the electric field strength E at height y above the midpoint between the lines. (LC, 2 point shot)
(in terms of y , d , and λ)

