15: Oscillatory Motion

One more time, we began Physics 181 with the observation that:

"Everything Moves"

And our goal has been to analyze and understand this motion. **Chapter 12 was devoted to the observation:**

"Some Things Rotate"

Now, in Chapter 15, we want to include the observation that:

"Some Things Wiggle"

Of course, a more technical term than wiggle is ***oscillate****!*

For example, swing something on a rope or cord, or take a meter stick, hold one end firmly clamped to a table, and with your other hand, push the other end down (not far enough to break it), and let it go.

Both of these are examples of damped oscillatory motion which we will get to.

Most of this chapter is devoted to developing a simple model of oscillating systems, **Simple Harmonic Motion**.

Simple Harmonic Motion

Some abbreviations that we'll use all the time:

SHM = **S**imple **H**armonic **M**otion
SHO = **S**imple **H**armonic **O**scillator

Important Question: In this context, what does **Harmonic** mean?

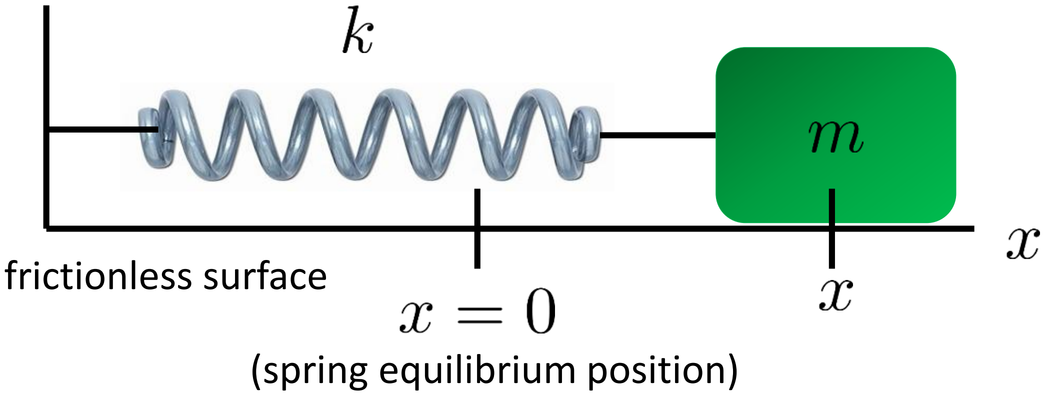
Mathematically, it means that the motion is describable in terms of **Harmonic Functions** which are **sines** and **cosines**.

In the first three sections, your author introduces SHM from an empirical point of view. He defines things like **period, frequency, angular frequency, amplitude**, the relation to uniform circular motion, and energy considerations.

Read these sections carefully. We're going to jump ahead to the dynamics treatment of SHM – *since we know how to do Newtonian dynamics*.
After that, we'll come back to some of the basic ideas.

SHM Dynamics

A useful Model of SHM is a mass on a linear spring on a frictionless surface. There are many many others, but we have worked with this system and know how to analyze it.



What happens when the mass is released?

PhET

As you saw on the PhET:

It oscillates between +x and –x. **What force causes the mass to go back and forth?**

Remember Hooke's Law for the Spring:

$$F_x = -kx \Rightarrow \begin{cases} F_x < 0 \text{ for } x > 0 \\ F_x > 0 \text{ for } x < 0 \end{cases}$$

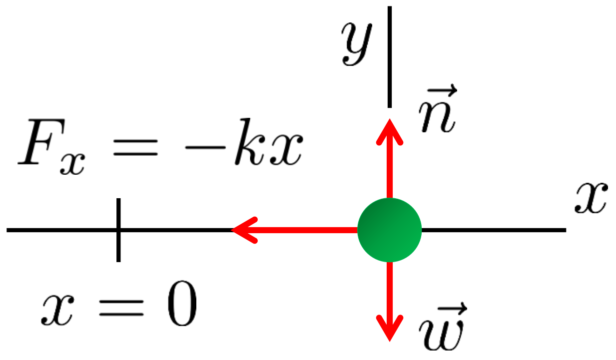
So the spring always pushes or pulls the mass back to x = 0, the equilibrium position of the spring.

This is known as a "**Linear Restoring Force**". **Linear** because the force is proportional to the first power of the displacement and **restoring** because the force always returns the system to its equilibrium position.

Important Point: Anytime you have a linear restoring force, you have simple harmonic motion.

SHM Dynamics: Obtaining the Equation

Free Body Diagram for m:



We know how to do this:

$$\sum F_y = n - w = ma_y = 0$$

$$\Rightarrow n = w = mg$$ *(Nothing real interesting here.)*

$$\sum F_x = -kx = ma_x$$

$$\Rightarrow a_x = -\frac{k}{m}x$$ *(Is this constant acceleration?)*

No, so to describe the motion of the block, **we have to solve:**

$$a_x = \boxed{\frac{d^2x}{dt^2} = -\frac{k}{m}x}$$

(What kind of animal is this?)

This is a second-order linear differential equation for the function x(t).

What do most people do when they're faced with a differential equation?

Don't worry – you probably haven't studied differential equations yet, so you won't be asked to solve one in HW or an exam – we'll walk through it together.

SHM Dynamics: Solving the Equation

So, our task is to solve the differential equation:

$$\boxed{\frac{d^2x}{dt^2} = -\frac{k}{m}x}$$

First: What is a differential equation?

"A differential equation is a mathematical relation between a function and some of the function's derivatives."

Second: What does it mean to solve a differential equation?

"Solving the differential equation means that you find the function that satisfies the relation."

Third: The mathematically sophisticated way to solve a differential equation is to:

Guess a function!

Fourth: Plug your guess into the differential equation and see if it's correct.

Fifth: If your guess works, how do we know it's the only solution?

Mathematicians have proved the Uniqueness Theorem which says that "linear differential equations have only one unique solution." So if our guess works, it's the only solution.

SHM Dynamics: Solving the Equation

So, for our equation, we are looking for a function ***x(t)*** whose second derivative is the negative of the function times some constants.

$$\boxed{\frac{d^2x}{dt^2} = -\frac{k}{m}x}$$

And, we remember from calculus that trigonometric functions have this property, e.g.

$$\text{For } a = \text{constant}, \ \frac{d}{dz}\sin az = a\cos az \quad \underline{\text{and}} \quad \frac{d}{dz}\cos az = -a\sin az$$

$$\text{So:} \ \frac{d^2}{dz^2}\cos az = \frac{d}{dz}(-a\sin az) = -a^2\cos az$$

So, here is our guess for the function that obeys the differential equation:

$$x(t) = A\cos(\omega t + \phi_0)$$

where: A, ω, and ϕ_0 are constants that we have to determine.

By the way, in our guess, there's a cosine; where's the angle?

There really isn't one. This is an example of using trig functions because of their oscillating nature. There's more to trig functions than right triangles.

Whiteboard Problem 15-1

Show by direct substitution that our guess:

$$x(t) = A\cos(\omega t + \phi_0)$$

is a solution of the differential equation (*just plug in the guess, and show that the left hand side is the same as the right had side*):

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

This should also show what one of the constants is.

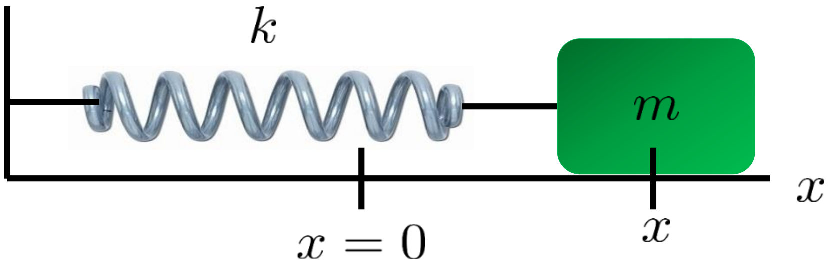
For $x(t)$ to be a solution, what must ω be? **LC**

For ***x(t)*** to be a solution, must have: $\omega = \sqrt{\frac{k}{m}}$

The other constants, A and ϕ_0, will be determined from the initial conditions.

SHM: The Basics

The model:



The differential equation:

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

The solution:

$$x(t) = A\cos(\omega t + \phi_0)$$

$$v(t) = \frac{dx}{dt} = -A\omega\sin(\omega t + \phi_0)$$

Note: be careful using these equations in your calculator. The argument of the sine and cosine is in radians, and your calculator has to be set to radians.

$$A = \text{Amplitude}$$

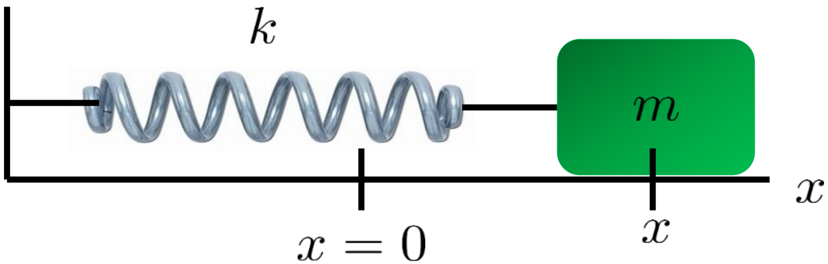
$$\omega = \sqrt{\frac{k}{m}} = \text{angular frequency } [\tfrac{rad}{s}]$$

$$f = \text{frequency } [\text{Hz} = \text{cycles/second}] = \tfrac{\omega}{2\pi}$$

$$T = \text{Period } [s] = \tfrac{1}{f} = \tfrac{2\pi}{\omega}$$

PhET

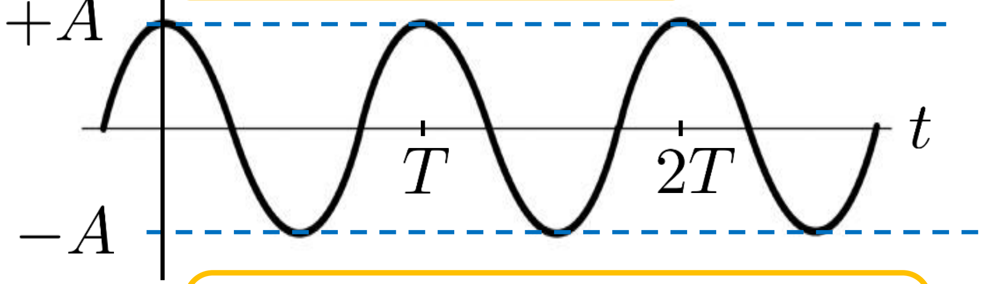
SHM: The Basics



If the block is pulled to $x = +A$ and released from rest at $t = 0$, then $A =$ the amplitude and $\phi_0 = 0$.

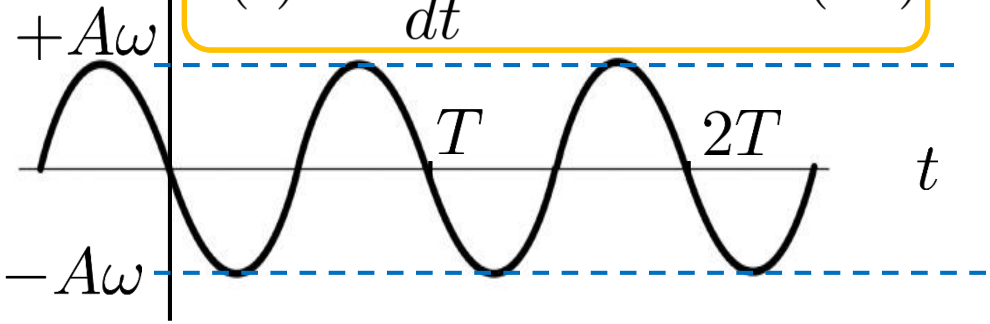
(If $x(t=0) \neq A$, then $\phi_0 \neq 0$ we'll deal with this ~~next class.~~ soon)

$$\boxed{x(t) = A\cos(\omega t)}$$



The position oscillates between $\pm A$ with period T

$$\boxed{v(t) = \frac{dx}{dt} = -A\omega\sin(\omega t)}$$



The velocity oscillates between $\pm\omega A$ with period T

Whiteboard Problem 15-2

An air track glider is attached to a spring and oscillates between the 10 cm mark and the 60 cm mark on the track. The glider completes 10 oscillations in 33 s.

Determine:

a) **the period and frequency (LC) in Hertz.**
b) **the angular frequency and the amplitude (LC)**
c) **the maximum speed of the glider (LC)**

Energy and SHM

Since the spring force is a conservative force, the mechanical energy of SHM is conserved:

$$\boxed{E = K + U_s = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \text{constant}}$$

Both the **kinetic energy (K)** and **potential energy (U_s)** change with time, but their sum, the **total energy, E, is constant.**

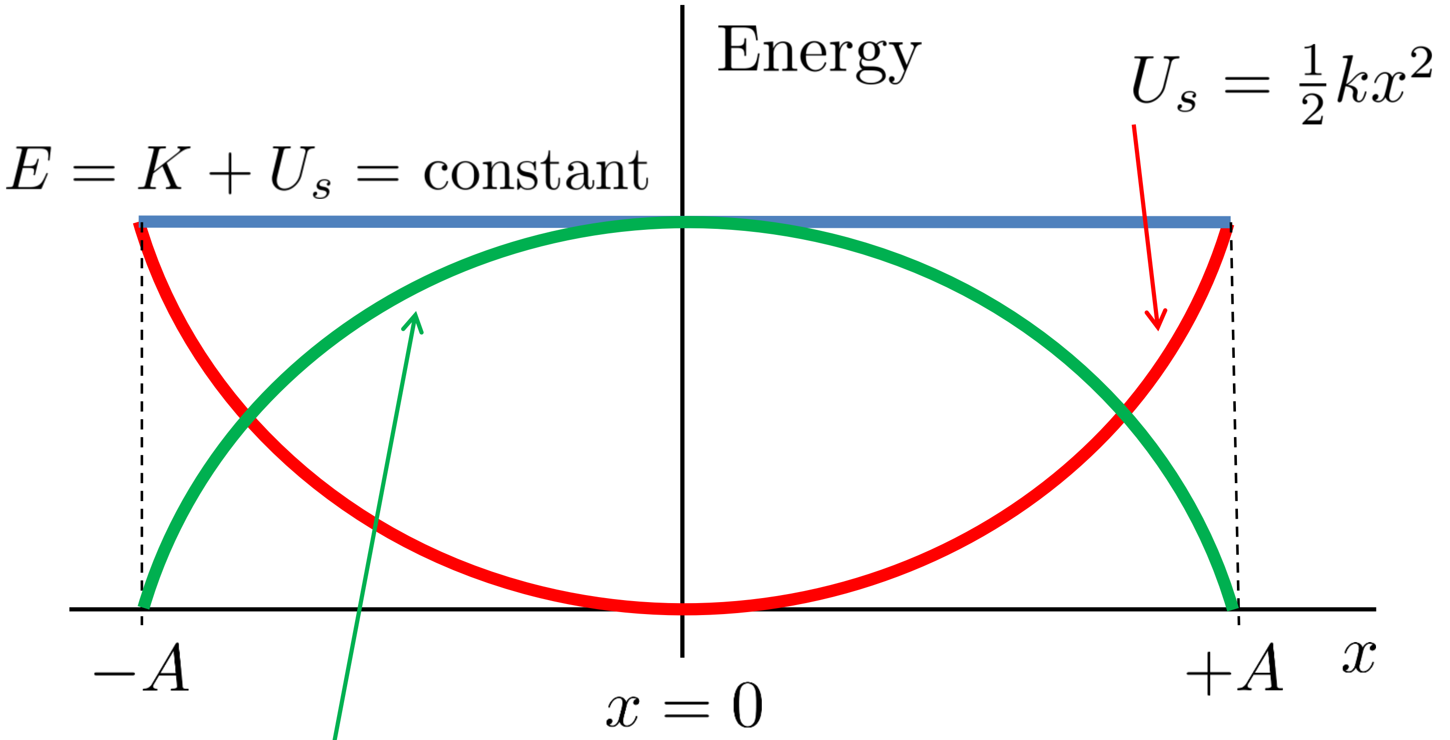
PhET

The above expression is for any time. There are two special times that give us **two valuable equations for the total energy**:

$$\text{At } x = \pm A, v = 0 \quad \Rightarrow \boxed{E = \tfrac{1}{2}kA^2}$$

$$\text{At } x = 0, v = \pm v_{\max} = \pm A\omega \quad \Rightarrow \boxed{E = \tfrac{1}{2}mv_{\max}^2 = \tfrac{1}{2}mA^2\omega^2}$$

SHM Energy Plots



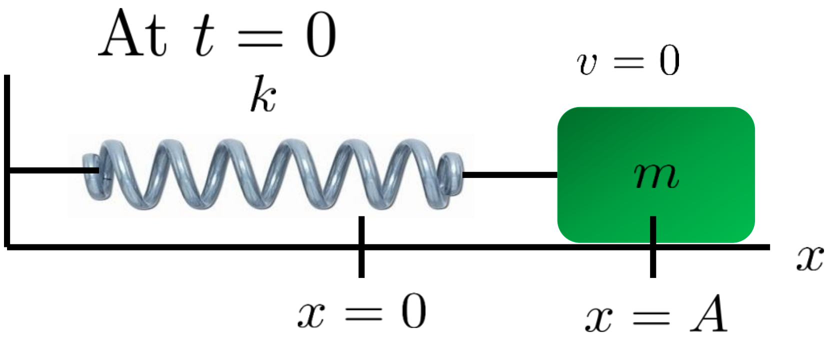
$$K = E - U_s = \tfrac{1}{2}kA^2 - \tfrac{1}{2}kx^2 = \tfrac{1}{2}k(A^2 - x^2)$$

$\pm A$ are called the turning points

PhET

Finding the Phase Constant from the Initial Conditions

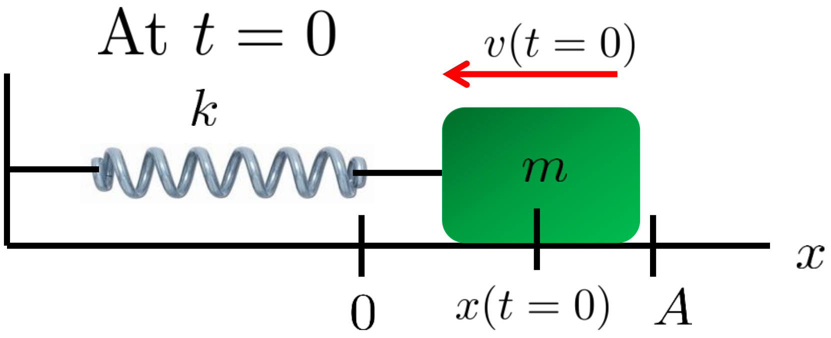
Recall what we had for the basic SHO:



If the block is pulled to $x = +A$ and released from rest at $t = 0$, then $A =$ the amplitude and $\phi_0 = 0$.

$$x(t) = A\cos(\omega t)$$

What if we have something like this for the initial conditions?



The block still oscillates between $\pm A$, but now $\phi_0 \neq 0$, so:

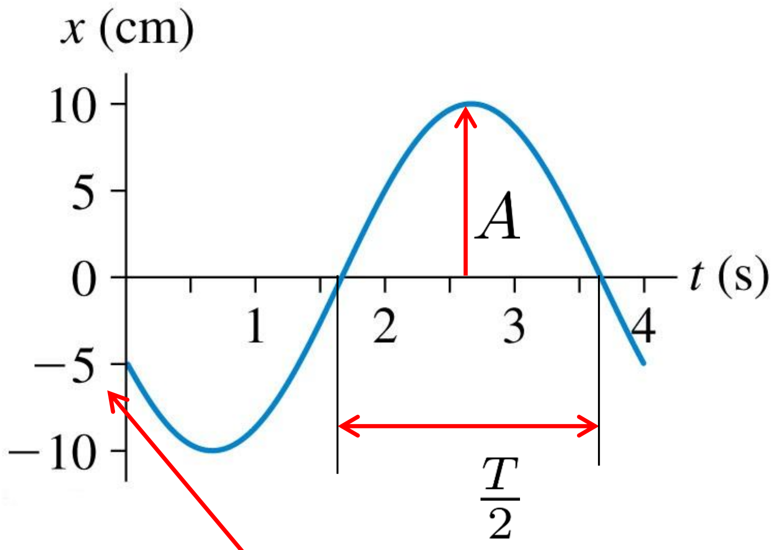
$$x(t) = A\cos(\omega t + \phi_0)$$

How can we use the initial position and velocity to find the phase constant?
This is best illustrated with in a problem

Example Whiteboard Prob: 15-3

The position vs. time graph for a particle in SHM is shown.

What is the phase constant, ϕ_0?



We can get some info from the graph:

$A = 10\text{ cm} \qquad \frac{T}{2} = 2\ s \Rightarrow T = 4\ s$

So, $f = \frac{1}{T} = \frac{1}{4}\ Hz \Rightarrow \omega = 2\pi f = \frac{\pi}{2}\ rad/s$

$$x(t) = A\cos(\omega t + \phi_0) \qquad v(t) = -A\omega\sin(\omega t + \phi_0)$$

To Find ϕ_0 : Set $t = 0$ $\quad x(0) = A\cos(\phi_0) = -5\ cm$ from the graph

$$\text{So, } \phi_0 = \cos^{-1}\left(\frac{-5}{10}\right) = \cos^{-1}(-.5)$$

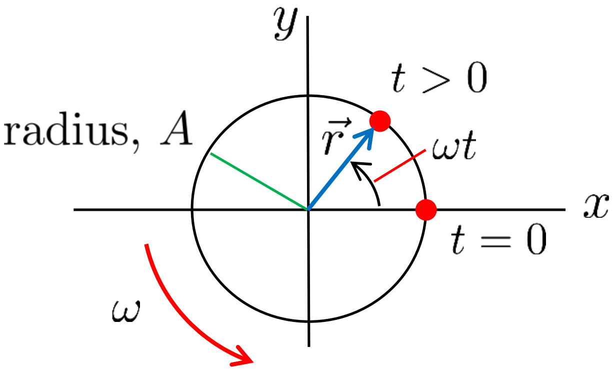
What does your calculator say?

$$\text{Actually, } \phi_0 = \pm\frac{2}{3}\pi\ rad\ (\pm 120^\circ)$$

Which one is it? *We need a way to figure this out.*

Finding the Phase Constant from the Initial Conditions

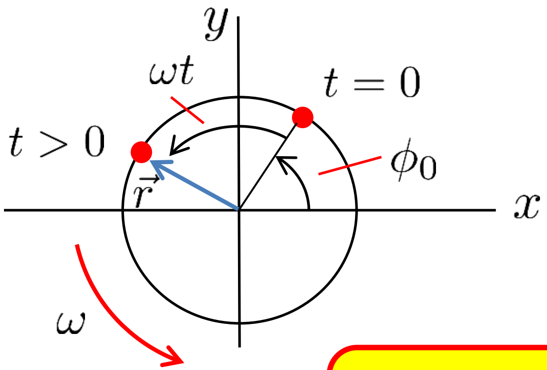
Your author presents a nice way to do this that is a **visualization tool based on uniform circular motion (UCM)**:



The x-component of the object's position vector is:

$$x(t) = A\cos(\omega t) \quad \text{Just SHM with } \phi_0 = 0!$$

What if the object started somewhere else?



Now, the x-component of the object's position vector is:

$$x(t) = A\cos(\omega t + \phi_0) \quad \text{SHM with } \phi_0 \neq 0!$$

So, ϕ_0 is the angle where the equivalent UCM object is at $t = 0$

Finding the Phase Constant from the Initial Conditions

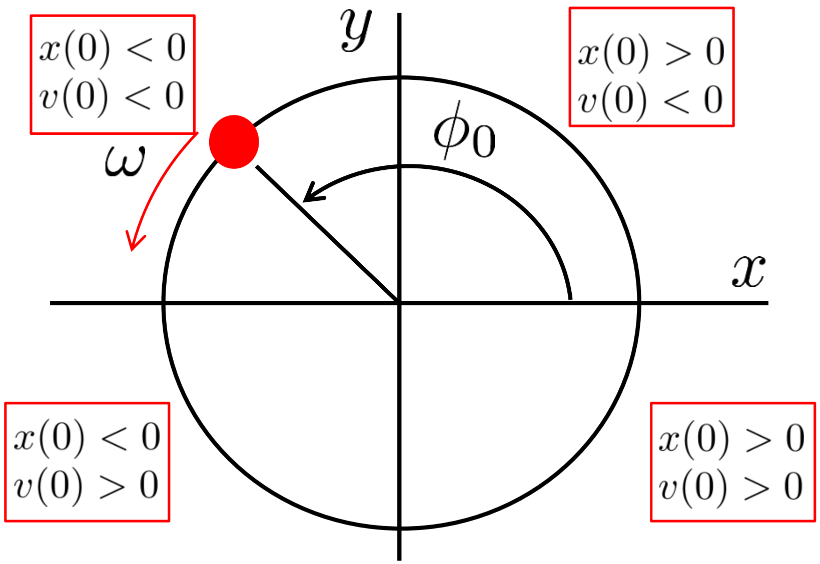
How do we use the UCM tool to find the phase constant?

We know the initial conditions: $x(0)$ and $v(0)$

$$x(t) = A\cos(\omega t + \phi_0) \qquad \Rightarrow x(0) = A\cos\phi_0$$

$$v(t) = -A\omega\sin(\omega t + \phi_0) \quad \Rightarrow v(0) = -A\omega\sin\phi_0$$

So, if we know x(0) and v(0), we can find what quadrant the equivalent UCM particle is in at t = 0. This gives the phase constant:



Let's see how this works for Example WB Prob 15-3

Example Whiteboard Problem 15-3 (what we had)

The position vs. time graph for a particle in SHM is shown.

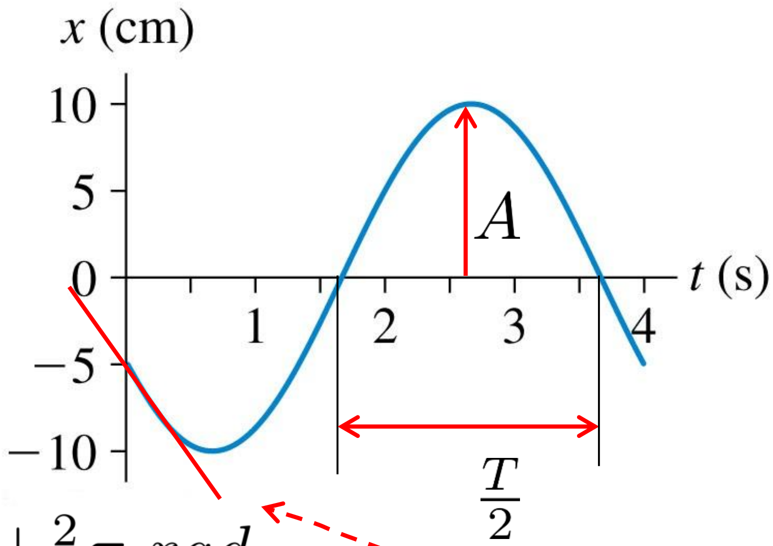
What is the phase constant, ϕ_0?

Previously, we found:

$A = 10\text{ cm} \quad T = 4\ s \quad \omega = \frac{\pi}{2}\ rad/s$

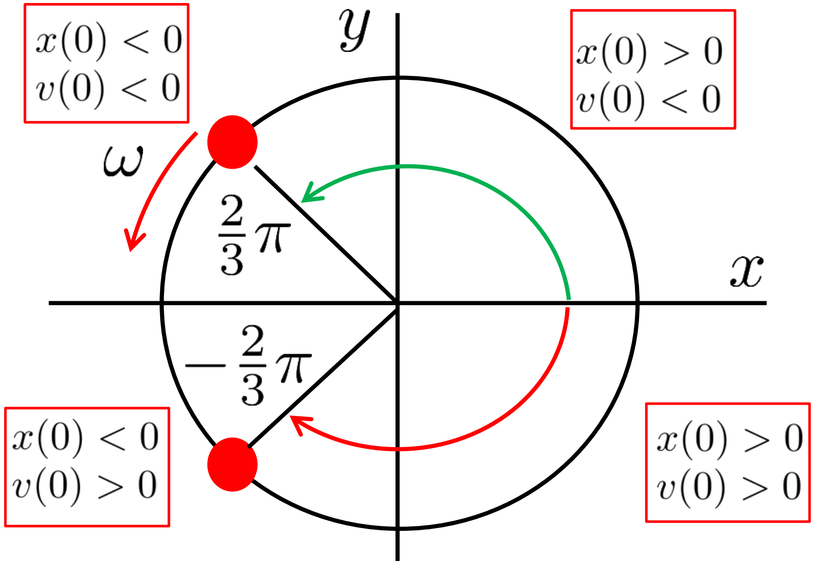
$$x(t) = A\cos(\omega t + \phi_0) \Rightarrow x(0) = A\cos\phi_0$$

$$v(t) = -A\omega\sin(\omega t + \phi_0)$$



slope at t = 0 < 0

$$\phi_0 = \cos^{-1}\left(\frac{-5}{10}\right) = \cos^{-1}(-.5) \Rightarrow \phi_0 = \pm\frac{2}{3}\pi\ rad$$



$x(0) < 0$
$v(0) < 0$

$x(0) > 0$
$v(0) < 0$

$x(0) < 0$
$v(0) > 0$

$x(0) > 0$
$v(0) > 0$

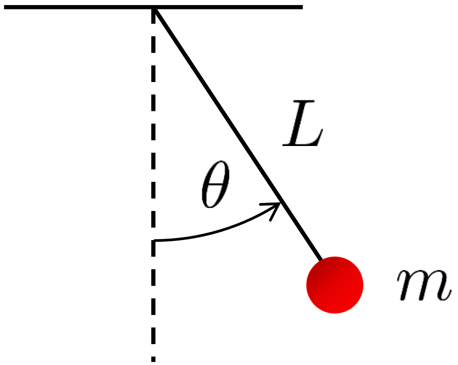
From the graph:

$$x(0) < 0 \text{ and } v(0) < 0$$

So, ϕ_0 must be in the second quadrant

Thus, $\phi_0 = +\frac{2}{3}\pi$ rad

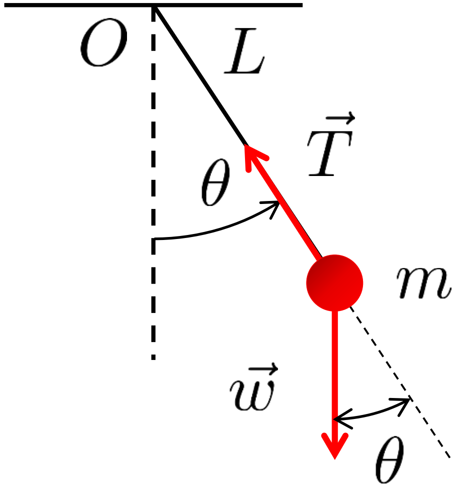
The Simple Pendulum



A simple pendulum is a point mass on a massless string or rod that can swing around a pivot point.

Your author has one way to show that this is SHM; here's another way:

Free Body Diagram:



$$\circlearrowleft + \quad \sum \tau_0 = -wL\sin\theta = I\alpha$$

$$-mgL\sin\theta = (mL^2)\frac{d^2\theta}{dt^2}$$

$$\text{Or, } \frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta$$

This is not SHM, but if we consider only <u>small angles</u>: $\sin\theta \approx \theta$

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta \quad \left[\text{Compare to } \frac{d^2x}{dt^2} = -\frac{k}{m}x, \text{ same DE} \Rightarrow \text{same sol'n}\right]$$

<u>So, for the simple pendulum</u>:

$$\omega = \sqrt{\frac{g}{L}} \qquad \text{and } T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}$$

$$\theta(t) = A\cos(\omega t + \phi_0)$$

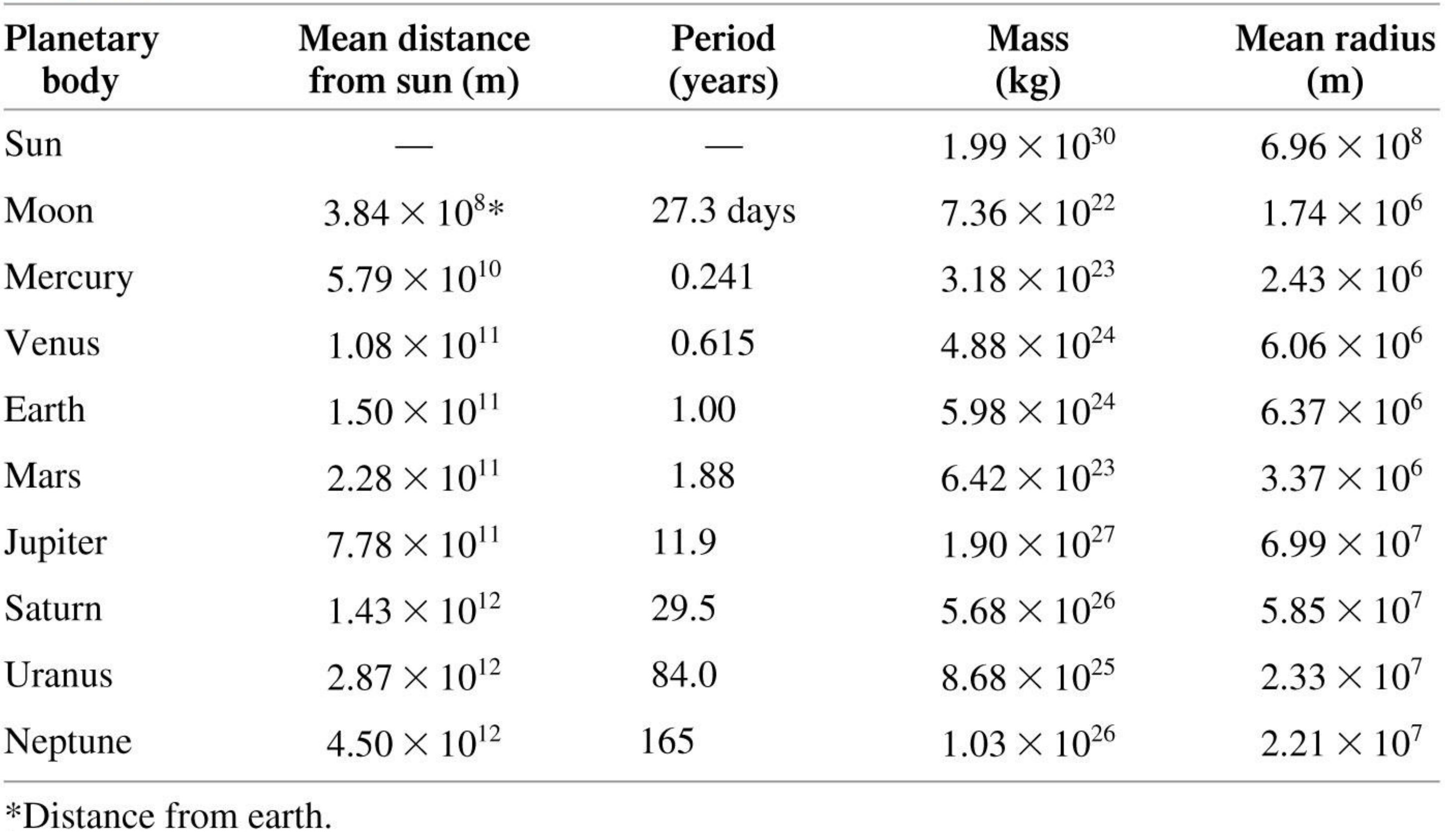
<u>PhET</u>

Whiteboard Problem 15-4

What is the period of a 1.0 m long pendulum on

- **a) the Earth? (LC)**
- **b) Venus? (LC)**

TABLE 13.2 Useful astronomical data **(you may need some of this data)**

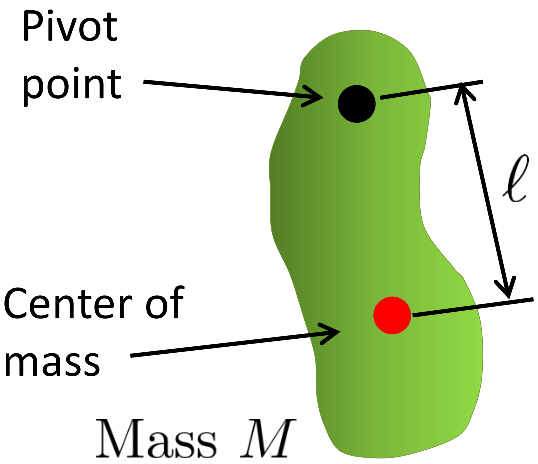


Planetary body	Mean distance from sun (m)	Period (years)	Mass (kg)	Mean radius (m)
Sun	—	—	1.99×10^{30}	6.96×10^{8}
Moon	3.84×10^{8}*	27.3 days	7.36×10^{22}	1.74×10^{6}
Mercury	5.79×10^{10}	0.241	3.18×10^{23}	2.43×10^{6}
Venus	1.08×10^{11}	0.615	4.88×10^{24}	6.06×10^{6}
Earth	1.50×10^{11}	1.00	5.98×10^{24}	6.37×10^{6}
Mars	2.28×10^{11}	1.88	6.42×10^{23}	3.37×10^{6}
Jupiter	7.78×10^{11}	11.9	1.90×10^{27}	6.99×10^{7}
Saturn	1.43×10^{12}	29.5	5.68×10^{26}	5.85×10^{7}
Uranus	2.87×10^{12}	84.0	8.68×10^{25}	2.33×10^{7}
Neptune	4.50×10^{12}	165	1.03×10^{26}	2.21×10^{7}

*Distance from earth.

The Physical Pendulum

A physical pendulum is a real object that can rotate about some pivot point:



Your author uses the same steps that we used for a simple pendulum to show that for small angles, this is also SHM where:

$$\omega = \sqrt{\frac{Mg\ell}{I_p}}$$

where: I_p = moment of inertia about the pivot point

$$= I_{\text{cm}} + M\ell^2 \quad \textit{(remember the parallel axis theorem?)}$$

So, the period is: $$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I_p}{Mg\ell}}$$

Whiteboard Problem 15-5

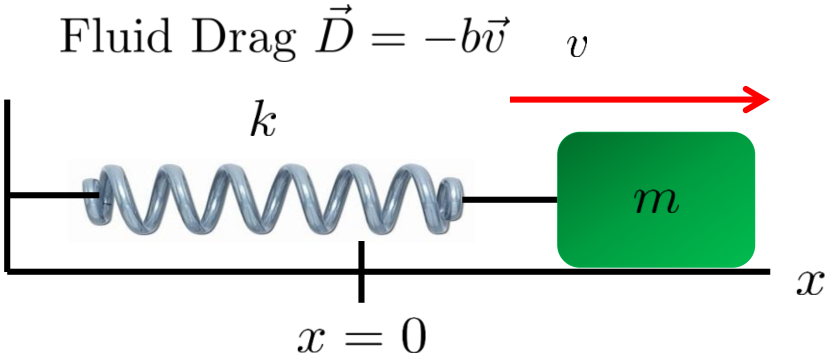
A uniform rod of mass M and length L swings as a pendulum on a pivot at a distance of L/4 from one end of the rod. **Find an expression for the frequency, *f*, for small angles. (LC)** (*your expression should contain only g, L, and numbers*.)

Damped Oscillations

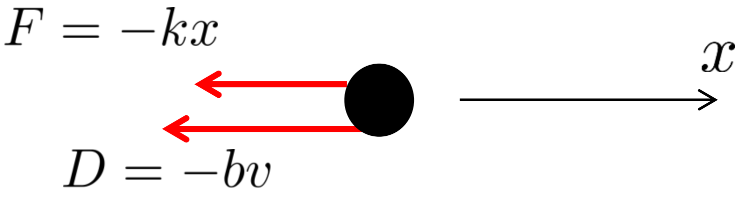
In real life, there is always some friction and the amplitude of a SHO will decrease with time. We call this **damping** – look at the mass spring systems on PhET.

PhET

Here's one model of a damped oscillator:



FBD:



b = damping constant

$$\sum F_x = -kx - bv = ma_x$$

$$\text{or,} \qquad -kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

$$\text{or,} \qquad \frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

This is not such an easy differential equation to guess a solution for; we'll just jump to the solution.

Damped Oscillations – the Solution

The solution for this model of a damped oscillator is:

$$x(t) = \underbrace{A_0 e^{-\frac{t}{2\tau}}}_{\text{decaying amplitude } A(t)} \underbrace{\cos(\omega t + \phi_0)}_{\text{oscillating part}}$$

$$\text{where } \tau = \frac{m}{b} = \textbf{time constant}$$

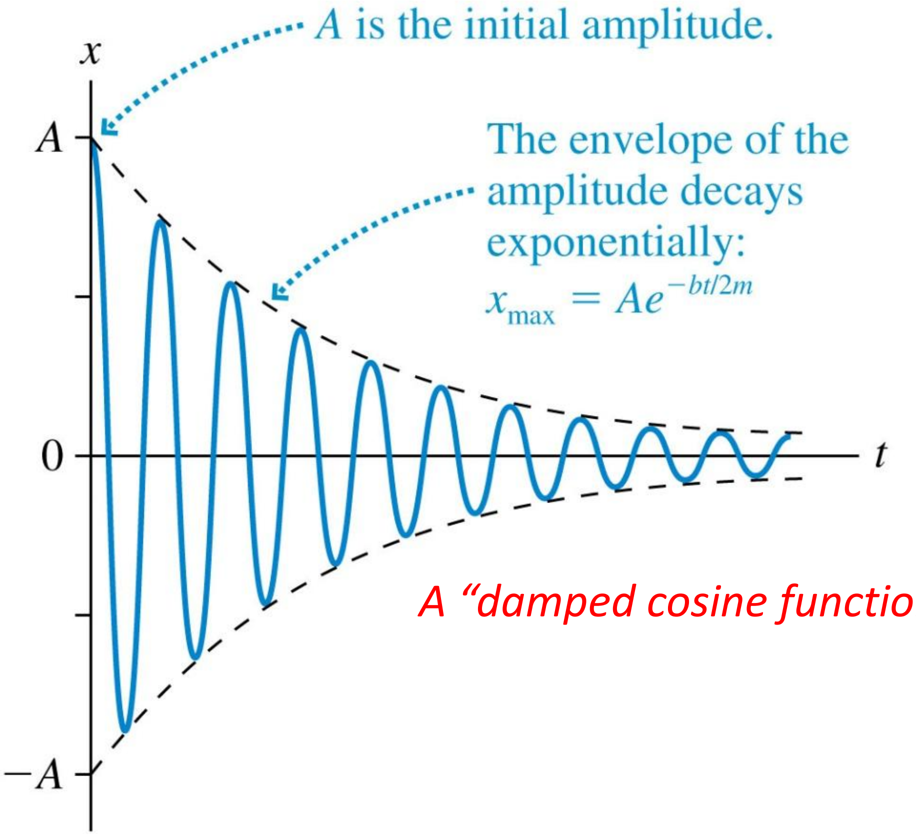
$$\omega = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

$$\omega = \sqrt{\omega_0^2 - \frac{1}{4\tau^2}}$$

$$\approx \omega_0 \ \text{(almost always)}$$

$$\text{Note: } \omega_0 = \sqrt{\frac{k}{m}}$$

The angular frequency of the undamped oscillator.



A "damped cosine function"

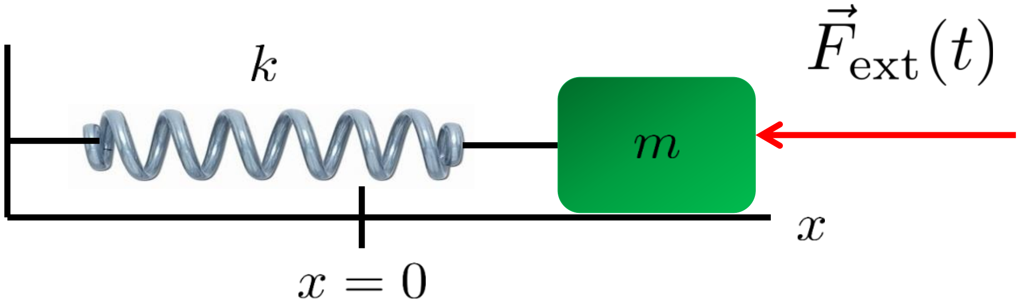
Whiteboard Problem 15-6

A 250 g air track glider is attached to a spring with spring constant 4.0 N/m. The damping constant due to air resistance is 0.015 kg/s (*this is the damping constant b*). The glider is pulled out 20 cm from equilibrium and released. **How many oscillations will it make during the time in which the amplitude decays to e^{-1} of its initial value? (LC)**

Hint: you just have to look at the decaying part: $A(t) = A_0 e^{-\frac{t}{2\tau}}$

Forced Oscillations and Resonance

What about an applied external force on an oscillator?



For most external forces, this is rather uninteresting, e.g. a constant force just shifts the equilibrium point.

But, a **periodic external force** can do some interesting things:

$$F_{\text{ext}}(t) \propto F_{\text{amp}} \cos(\omega_{\text{ext}} t) \qquad (\omega_{\text{ext}} = \text{angular frequency of the applied force})$$

With no damping, the solution is of the form:

$$x(t) \approx \frac{A}{|\omega_0 - \omega_{\text{ext}}|} \cos(\omega_{\text{ext}} t + \phi_0) \qquad \omega_0 = \sqrt{\frac{k}{m}} = \text{“natural frequency”}$$

So, for ω_{ext} not near $\omega_0 \Rightarrow$ small amplitude

But, for $\omega_{\text{ext}} \approx \omega_0 \Rightarrow$ Huge amplitude! **This is called Resonance**

Some Examples of Resonance

(links to videos work)

A forced oscillator showing resonance – this kid, like any kid, really understands resonance!

Resonant Sound Vibrations

You can try this yourself – wear protective glasses!

A famous example of an unplanned resonance – The Tacoma Narrows Bridge

This is the new bridge that was designed correctly.