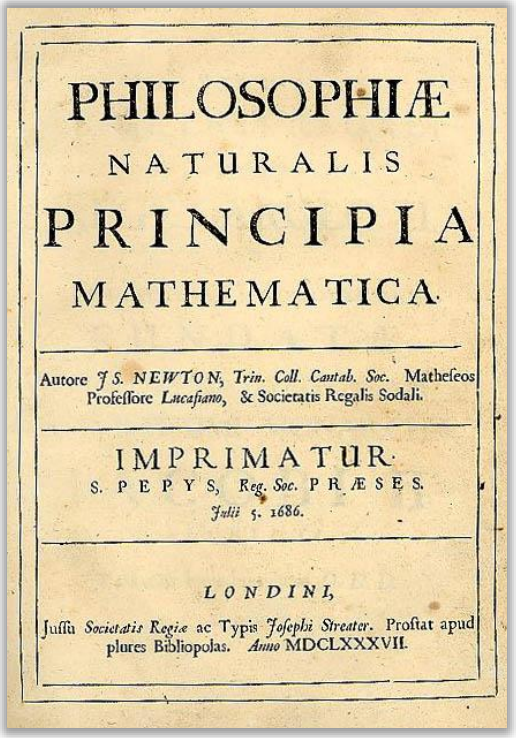
13: Newton's Theory of Gravity

After our first exam, we saw the NOVA video, ***Newton's Dark Secrets***.

There we saw how Newton arrived at his **Three Laws of Motion** that we have been studying and using these past several weeks.


PHILOSOPHIÆ
NATURALIS
PRINCIPIA
MATHEMATICA.

Autore *J. S. NEWTON*, *Trin. Coll. Cantab. Soc.* Matheseos Professore *Lucasiano*, & Societatis Regalis Sodali.

IMPRIMATUR.
S. PEPYS, *Reg. Soc.* PRÆSES.
Julii 5. 1686.

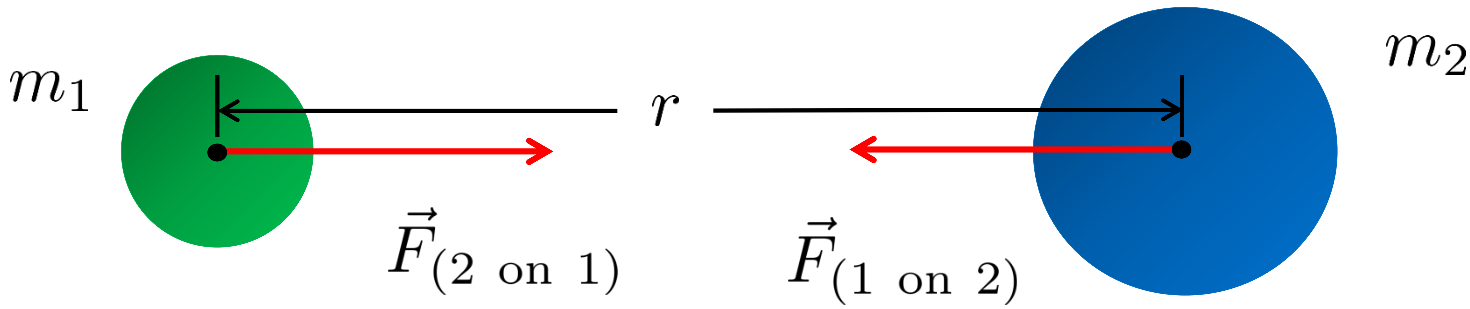
LONDINI,

Jussu *Societatis Regiæ* ac Typis *Josephi Streater*. Prostat apud plures Bibliopolas. *Anno* MDCLXXXVII.

The other part of **The Principia** was **Newton's Theory of Gravity.** It was here that Newton finally settled the question of *The Motion of the Planets*, and opened the way for us to move beyond the Earth.

A very brief account of the history of our struggle to understand the motion of the planets is in your text, read it carefully; we won't cover it in class.

Newton's Theory of Gravity



$$\text{Force } \underline{\textbf{Magnitudes}}\text{:} \quad F_{(2\ on\ 1)} = F_{(1\ on\ 2)} = G\frac{m_1 m_2}{r^2}$$

(Have to put in the direction by hand)

Note: This is an **inverse square force** – i.e. force magnitude falls off as 1/distance^2 that's what gives elliptical orbits.

The gravitation force is **always attractive** – this gives the directions.

In addition, the force is **along a line between the centers of the objects**.

(in fact, it is really for particles only, but Newton showed that spherically symmetric objects can be treated as point particles with all of the mass concentrated at the center.)

$$G = \text{Gravitation Constant} = 6.67 \times 10^{-11}\ \frac{Nm^2}{kg^2}\ \text{(in MKS units)}$$

(where to store this in your calculator?)

Whiteboard Problem 13-1

What is the ratio of the Sun's gravitational force on the Moon to the Earth's gravitational force on the Moon? **(LC)**

Hint: instead of calculating lots of numbers, think in terms of ratios.

TABLE 13.2 Useful astronomical data ***(you will need some of this data)***

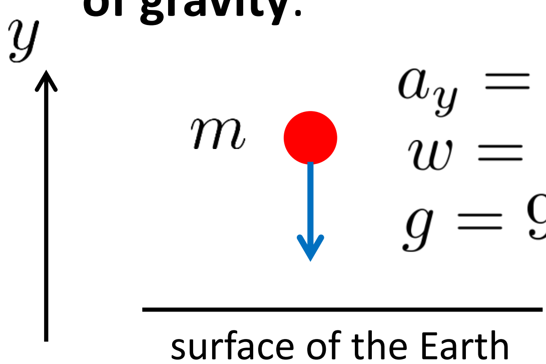
Planetary body	Mean distance from sun (m)	Period (years)	Mass (kg)	Mean radius (m)
Sun	—	—	1.99×10^{30}	6.96×10^{8}
Moon	3.84×10^{8}*	27.3 days	7.36×10^{22}	1.74×10^{6}
Mercury	5.79×10^{10}	0.241	3.18×10^{23}	2.43×10^{6}
Venus	1.08×10^{11}	0.615	4.88×10^{24}	6.06×10^{6}
Earth	1.50×10^{11}	1.00	5.98×10^{24}	6.37×10^{6}
Mars	2.28×10^{11}	1.88	6.42×10^{23}	3.37×10^{6}
Jupiter	7.78×10^{11}	11.9	1.90×10^{27}	6.99×10^{7}
Saturn	1.43×10^{12}	29.5	5.68×10^{26}	5.85×10^{7}
Uranus	2.87×10^{12}	84.0	8.68×10^{25}	2.33×10^{7}
Neptune	4.50×10^{12}	165	1.03×10^{26}	2.21×10^{7}

*Distance from earth.

This is why we say that the Moon really orbits the Sun with the Earth perturbing its path: PhET

The Acceleration of Gravity: *"Little g" & "Big G"*

Since Chap 2, we have been using **this approximation for the acceleration of gravity**:

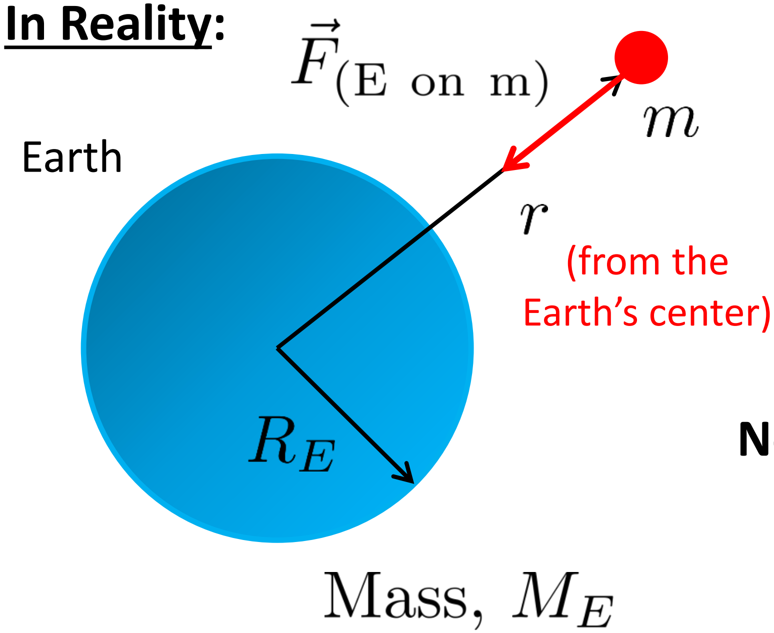


$$a_y = -g$$
$$w = mg$$
$$g = 9.8\ \frac{m}{s^2}$$

This is true <u>only</u> near the surface of the Earth!

$$F_{(\mathrm{E\ on\ m})} = \frac{G M_E m}{r^2} = ma$$

<u>In Reality</u>:



So, the acceleration of gravity really is:

$$a = g(r) = \frac{G M_E}{r^2}$$

(we did something odd here, does anyone know what?)

(We cancelled m, but one is a gravitational mass and the other is an inertial mass; who says that they're the same? See the Principal of Equivalence on p. 346 of your text.)

Note:

$$g(r = R_E) = \frac{G M_E}{R_E^2} = 9.8\ \frac{m}{s^2}$$

Whiteboard Problem 13-2

Planet Z is 10,000 km in diameter. The free-fall acceleration on the surface of Planet Z is 8.0 m/s^2.

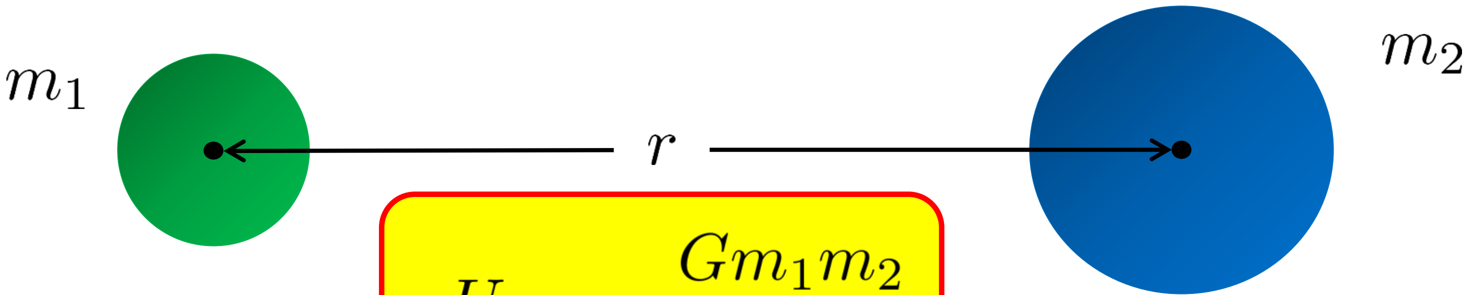
a) **What is the mass of Planet Z? (LC)**
b) **What is the free-fall acceleration 10,000 km above Planet Z's north pole? (LC)**

Note: the 10,000 km is an altitude above the surface of the planet. The r in the force and acceleration equations has to be measured from the center of the planet.

Gravitational Potential Energy

So: $g = 9.8\ m/s^2$ and weight, $w = mg$, **only work near the surface of the Earth**.
The same is true for our expression for **Gravitational Potential Energy, $U_g = mgy$.**

Your author derives a New Expression for the Gravitational Potential Energy that works anywhere:



$$U_g = -\frac{Gm_1m_2}{r}$$

Note: $U_g \longrightarrow 0$ for $r \longrightarrow \infty$ (that's where the zero is defined)

The negative sign is important, it tells us gravity is attractive.

$U_g \propto 1/r$ (Not $1/r^2$!) *(I have trouble remembering this one!)*

We should use this form of the gravitational potential energy in conservation of energy when r changes significantly.

Whiteboard Problem 13-3: Escape Velocity

The escape velocity is the minimum speed needed to escape from the surface of a planet and leave its gravitational influence.

a) **Use conservation of energy to find an expression for the minimum initial speed for the rocket to escape to infinity. (LC)** *(ignore any effects of the atmosphere)*

b) **Calculate the escape speed from the surface of the Earth (LC) (M_E = 5.98 X 10^{24} kg; R_E = 6.37 X 10^6 m)**

Circular Orbits for m << M; a planet and Star

A circular orbit is an application of uniform circular motion:

We can find an expression for the constant speed:

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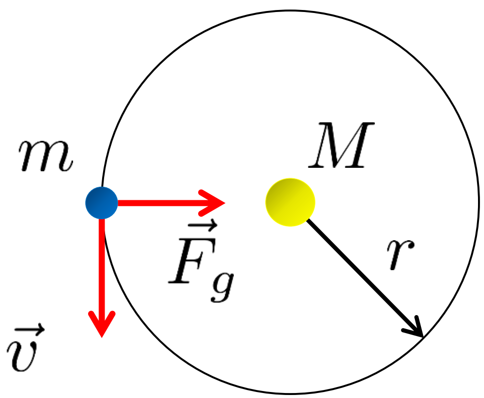
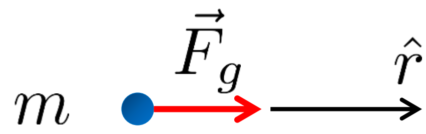


Free Body Diagram of m:



For Uniform Circular Motion:

$$\sum F_r = F_g = ma_r = \frac{mv^2}{r}$$

$$\text{So,}\ \frac{GmM}{r^2} = \frac{mv^2}{r}$$

Therefore, we have the **circular orbit speed**:

$$v_{\text{circ}} = \sqrt{\frac{GM}{r}}$$

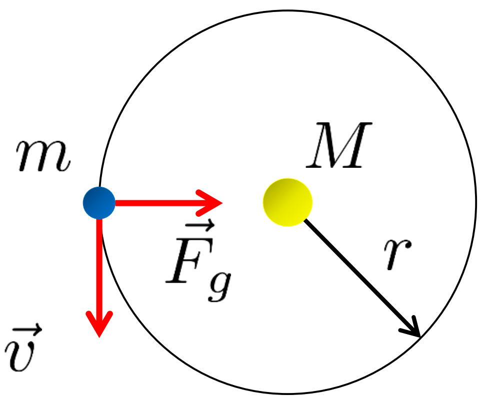
Circular Orbits for m << M; a planet and Star

Remember Kepler's Third Law from reading your text? ***The square of the orbital period (T) is <u>proportional</u> to the cube of the semimajor axis (a) of the orbit.*** **Newton could calculate this as an <u>equality</u>.**

For a <u>circular orbit where m << M</u>, we can obtain an important result:

$$\text{Period, } T = \frac{\text{distance}}{\text{speed}} = \frac{2\pi r}{v}$$
$$= 2\pi r\sqrt{\frac{r}{GM}}$$

$$\text{Or, } T^2 = 4\pi^2 r^2 \frac{r}{GM}$$



So, we have **<u>Newton's Form of Kepler's Third Law for m << M</u>**:

$$T^2 = \left(\frac{4\pi^2}{GM}\right) r^3$$

This equation can also be used for an elliptical orbit if you replace the radius, r, with the semimajor axis, a.

The above equations work when m << M. If that is not the case, both masses orbit about their common center of mass – see next WB.

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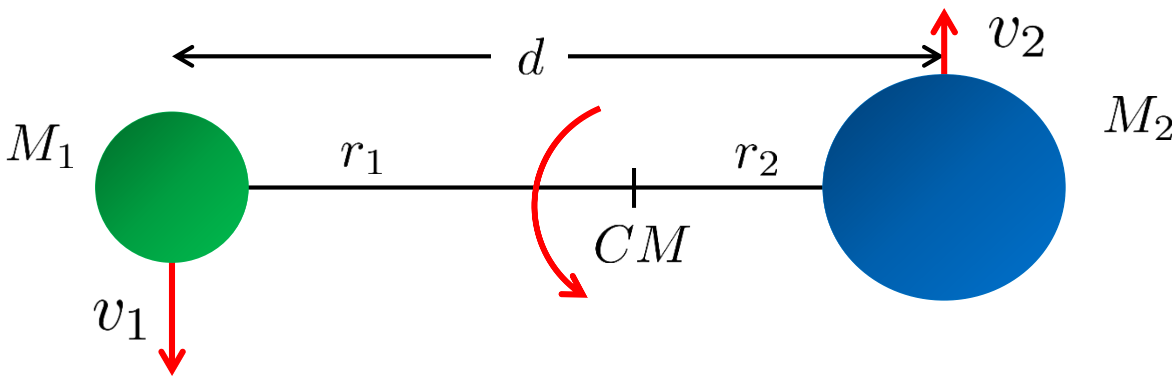
Whiteboard Problem 13-4: Binary Stars

Consider two stars with masses M_1 and M_2 separated by a distance d and orbiting their center of mass in circular orbits.

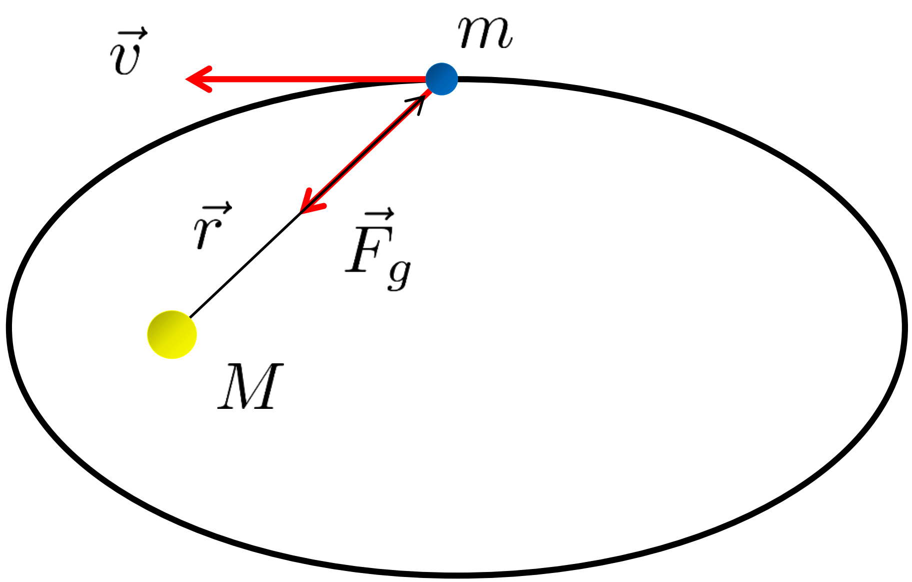


a) Find an expression for the orbital speed of M_1 in terms of M_1, M_2, r_1, and r_2. **(LC)**

For this situation, M_1 and M_2 have comparable masses, so the orbit equations for m << M won't work. Start at the beginning, e.g. a FBD of M_1.

b) Without doing any work, what is the speed of M_2?

Elliptical Orbits for m << M



For an elliptical orbit, the force varies, and the planet speeds up and slows down – that's Kepler's Second Law.

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But, we know that for any orbit, some things are conserved.

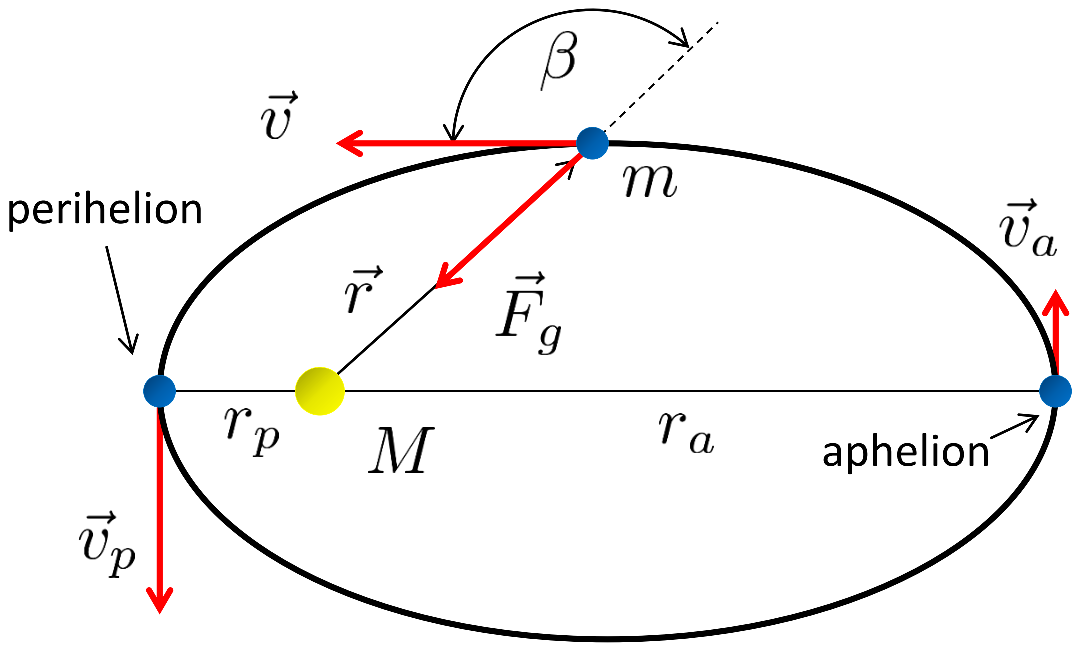
Since there are no nonconservative forces acting on m, the **mechanical energy is conserved**.

$$E_{\text{mech}} = \frac{1}{2}mv^2 - \frac{GMm}{r} = \text{constant}$$

This can be used to connect the speeds and distances for any two points in the orbit:

$$\frac{1}{2}v_1^2 - \frac{GM}{r_1} = \frac{1}{2}v_2^2 - \frac{GM}{r_2}$$

Elliptical Orbits for m << M



Note that, since the force of gravity is always directed toward the center, the torque on m about M is always zero, i.e.

$$\vec{\tau} = \vec{r} \times \vec{F}_g = 0$$

So, the angular momentum is conserved.

$$|\vec{L}| = |\vec{r} \times \vec{p}| = |\vec{r} \times m\vec{v}| = mrv \sin\beta \quad = \text{constant}$$

This can be cumbersome to use at an arbitrary point in the orbit, but if we look at just the **perihelion** point and the **aphelion** points where the angle is 90°:

$$L_p = L_a \quad \Longrightarrow r_p v_p = r_a v_a$$

Whiteboard Problem 13-5

The dwarf planet Pluto moves in a fairly elliptical orbit. At its closest approach to the Sun of 4.43 X 10^9 km (perihelion), Pluto's speed is 6.12 km/s.

What is Pluto's speed at its most distant point in its orbit (aphelion), 7.30 X 10^9 km? (LC)

Pluto from the New Horizons Spacecraft in 2015

Hint: **Draw the orbit and the points mentioned. There are two ways to do this problem. They both work, but one is a lot easier.**