12-2: Rotational Motion – Part 2

Now that we know about **rotational kinematics** and how to calculate **torques**, we can **begin to look at rotational dynamics problems**. The central idea is:

Rotational Dynamics:

A net torque on an object causes an angular acceleration. Or:

$$\alpha = \frac{\tau_{\text{net}}}{I} \quad \text{or} \quad \tau_{\text{net}} = I\alpha$$

This is Newton's 2nd Law for Rotation. But what is I?

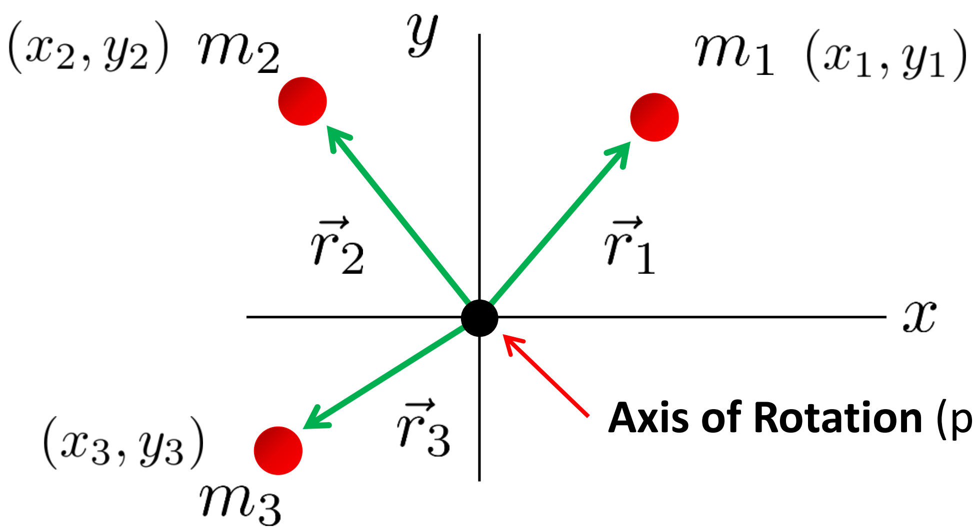
$$I = \text{Moment of Inertia, Units} = [\text{kg m}^2]$$

"Rotational Analog of Mass"

The moment of inertia of an object is determined by its mass **<u>and</u>** how the mass is distributed about the axis of rotation.

Moment of Inertia

Collection of Discrete Point Masses at fixed positions:



Axis of Rotation (perpendicular to x-y plane)

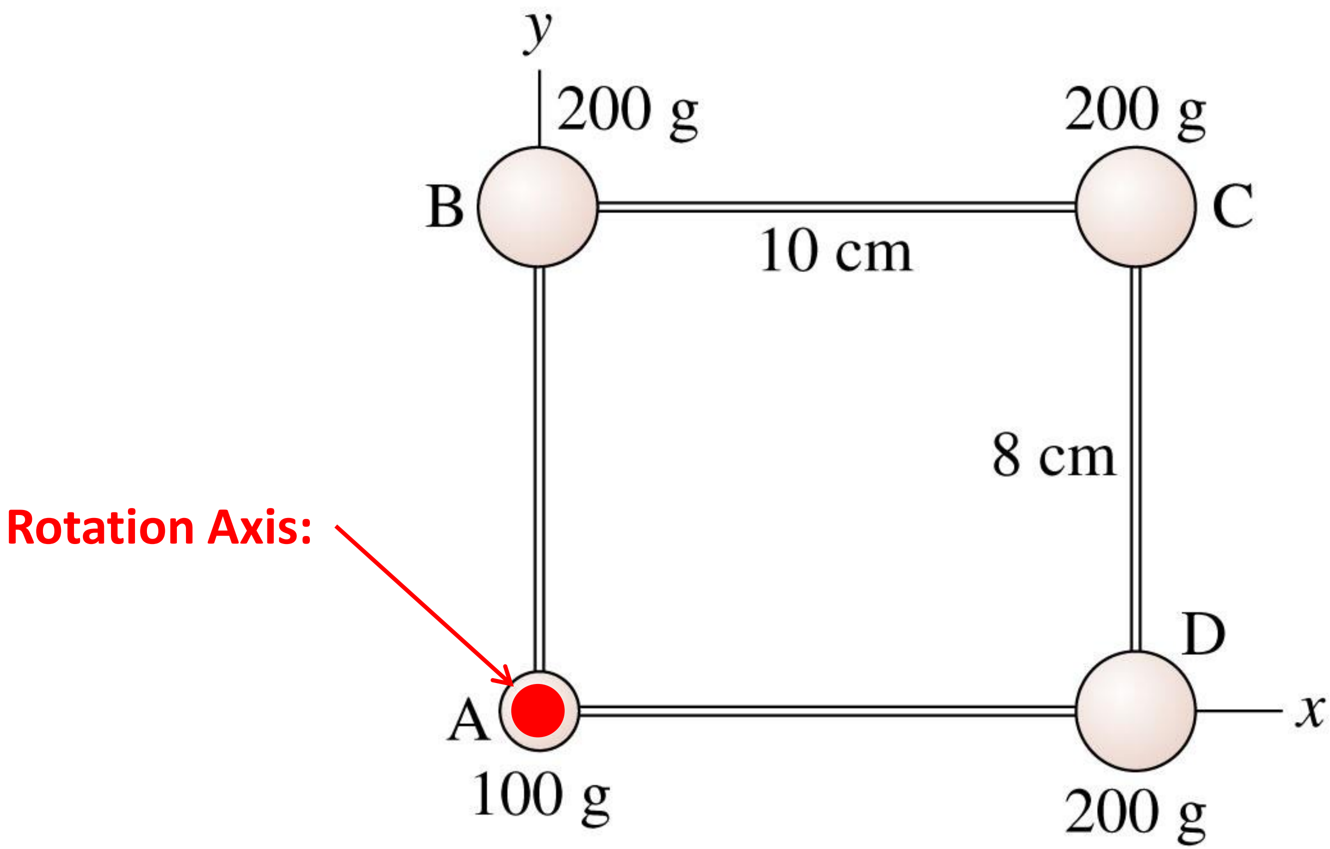
$$\text{Moment of Inertia, } I = \sum_{i=1}^{N} m_i r_i^2$$

$$\text{where: } r_i = \text{perpendicular distance from the axis to } m_i$$

$$= |\vec{r}_i| = \sqrt{x_i^2 + y_i^2}$$

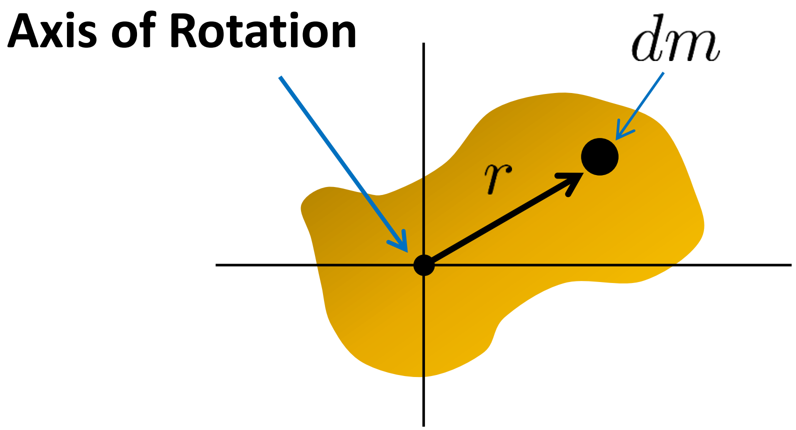
Whiteboard Problem 12-7

Find the moment of inertia of the four masses below about an axis that passes through mass A and is perpendicular to the page. (LC)



Moment of Inertia

Continuous Bodies:

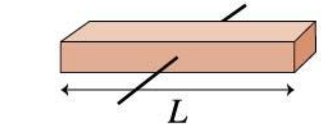
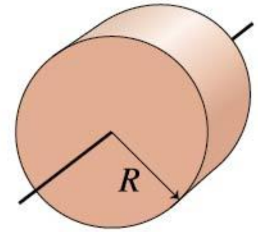
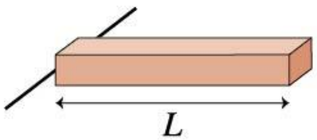
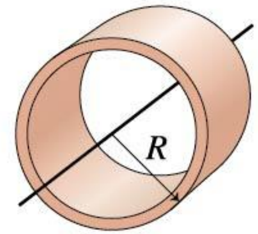
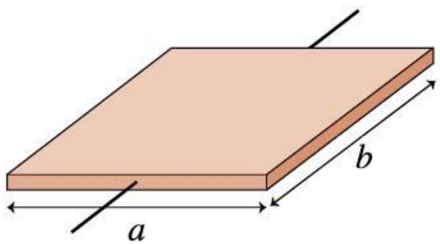
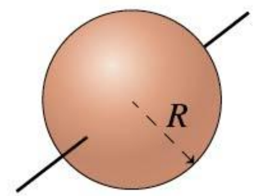
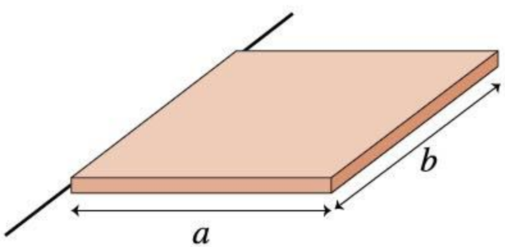
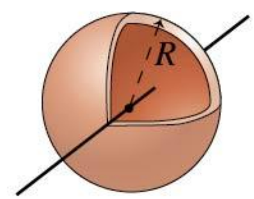


$$\text{Moment of Inertia, } I = \int_{\text{Volume}} r^2 dm$$

These can be difficult integrals. **We won't do any***, but we will use ones already calculated:

**If you are an engineering major, you'll do these in an engineering mechanics class.*

TABLE 12.2 Moments of inertia of objects with uniform density

Object and axis	Picture	I	Object and axis	Picture	I
Thin rod, about center		$\frac{1}{12}ML^2$	Cylinder or disk, about center		$\frac{1}{2}MR^2$
Thin rod, about end		$\frac{1}{3}ML^2$	Cylindrical hoop, about center		MR^2
Plane or slab, about center		$\frac{1}{12}Ma^2$	Solid sphere, about diameter		$\frac{2}{5}MR^2$
Plane or slab, about edge		$\frac{1}{3}Ma^2$	Spherical shell, about diameter		$\frac{2}{3}MR^2$

Note: you have to specify the axis of rotation too!

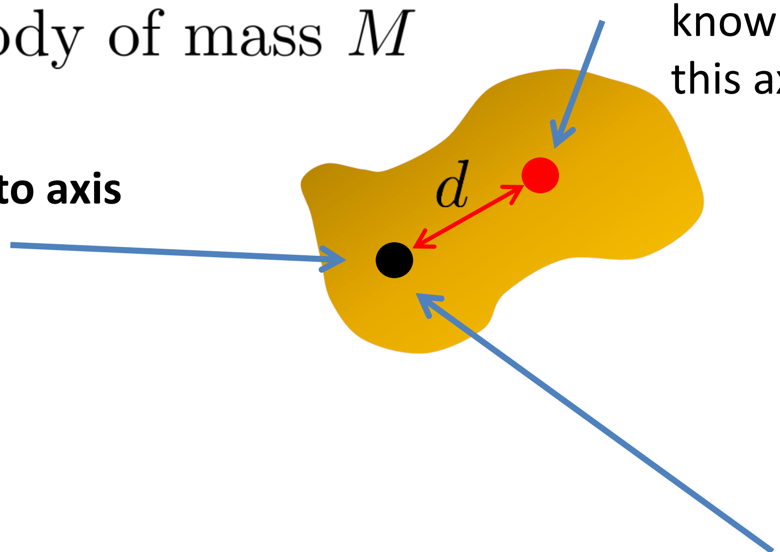
Parallel Axis Theorem

If you know the moment of inertia about an axis through the object's center of mass, you can find the moment of inertia about any parallel axis.

body of mass M

Axis through the center of mass, know the moment of inertia about this axis, (I_{cm})

Axis parallel to axis through CM



The moment of inertia about this parallel axis is:

$$I = I_{cm} + Md^2$$

where d is the distance between the axes

PROBLEM-SOLVING STRATEGY 12.1

MP

Rotational dynamics problems (where the rotation is about a fixed axis)

MODEL Model the object as a rigid body.

VISUALIZE Draw a pictorial representation to clarify the situation, define coordinates and symbols, and list known information.

- Identify the axis about which the object rotates.
- Identify forces and determine their distances from the axis. For ~~most~~ All problems it will be useful to draw a free-body diagram.
- Identify any torques caused by the forces and the signs of the torques.

SOLVE The mathematical representation is based on Newton's second law for rotational motion:

$$\tau_{net} = I\alpha \quad \text{or} \quad \alpha = \frac{\tau_{net}}{I}$$

- Find the moment of inertia in Table 12.2 or, if needed, calculate it as an integral or by using the parallel-axis theorem.
- Use rotational kinematics to find angles and angular velocities.

ASSESS Check that your result has correct units and significant figures, is reasonable, and answers the question.

Exercise 28

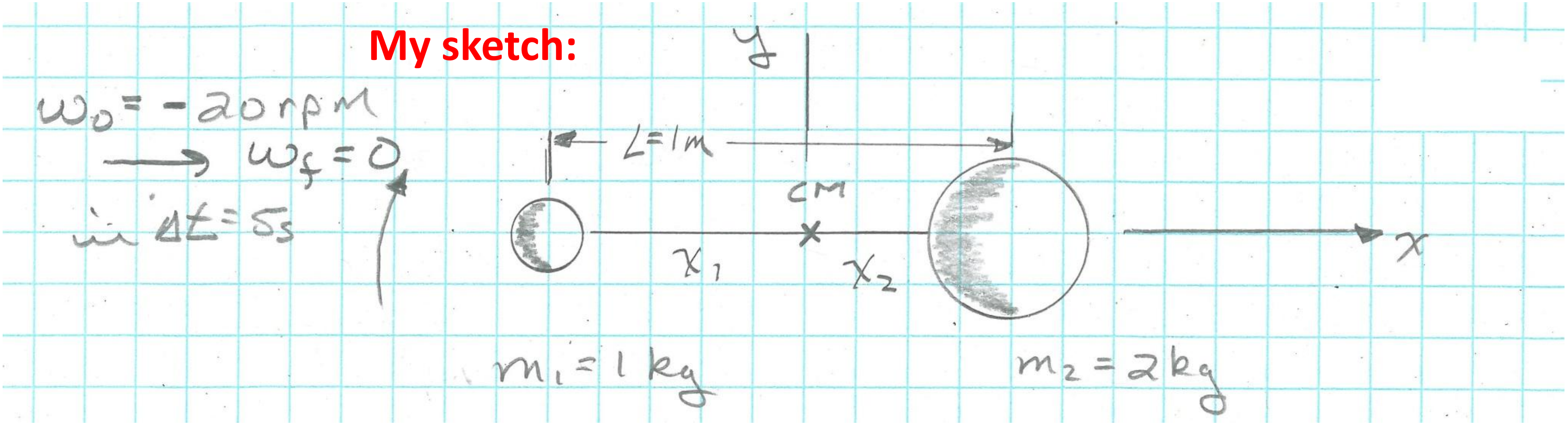
Whiteboard Problem 12-8

A 1.0 kg ball and a 2.0 kg ball are connected by a 1.0 m long rigid, massless rod to form a *dumbbell*. The *dumbbell* is rotating CW about its center of mass at 20 rpm.

a) **Find the location of the center of mass, *i.e. the distance x_1 in the sketch*.** **(LC)**

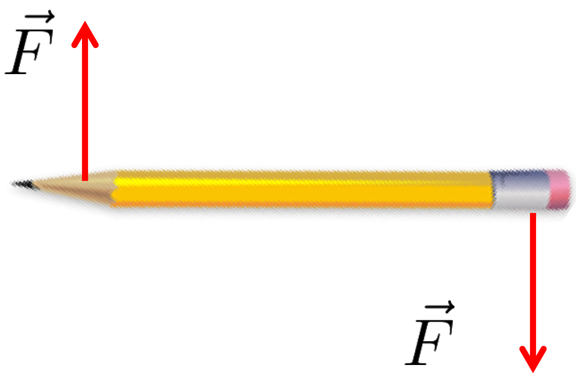
b) **Find the moment of inertia about the CM. (LC)**

c) **Now, find the constant torque that will bring the balls to rest in a time of 5.0 s. (LC)**

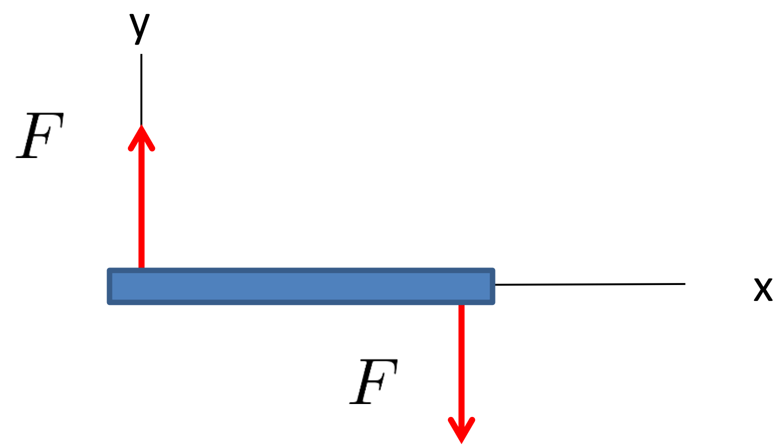


Static Equilibrium: *Remember this slide from Chapter 6?*

Static Equilibrium for objects that can't be represented as a particle is somewhat different: consider a pencil subject to the two forces:



FBD:

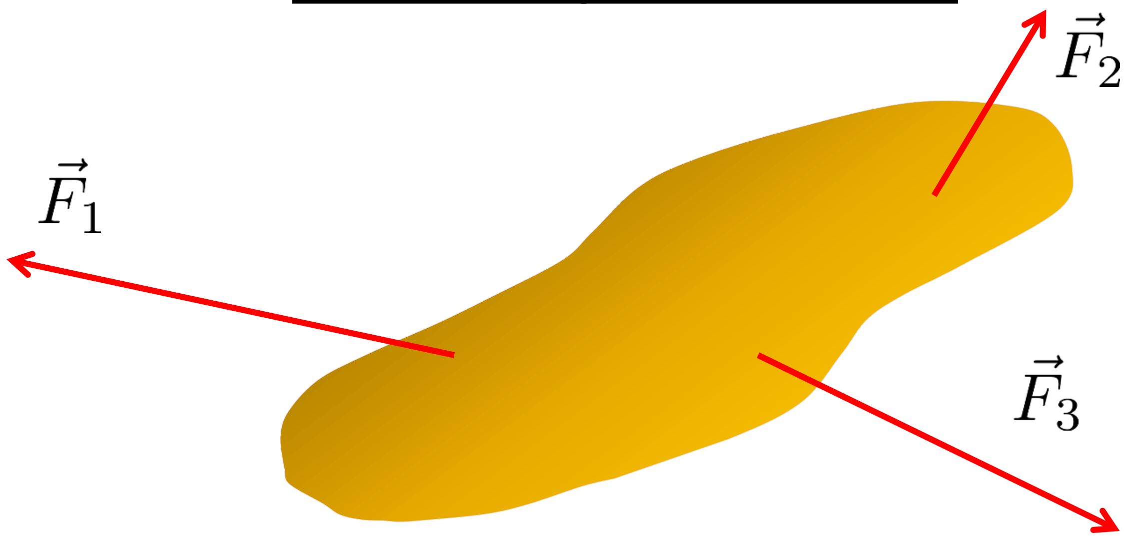


$$\sum F_x = 0 \text{ and } \sum F_y = 0, \text{ but is the pencil in equilibrium?}$$

No, it will rotate with an angular acceleration. For real objects in equilibrium, you need the force components to be zero and the sum of the torques about any point to also be zero - **don't worry now, we'll do this in Chapter 12**.

It's time treat this now:

Static Equilibrium*



The body is in static equilibrium, if:

$$\vec{F}_{\text{net}} = 0$$

$$\vec{\tau}_{\text{net}} = 0$$

Use in Component Form:

$$\sum F_x = 0 \qquad \sum F_y = 0$$

$$\sum F_z = 0 \;\; \text{(if necessary)}$$

$$\sum_{\text{(any point)}} \tau(\text{from all } \vec{F}_i) = 0$$

The sum of the torques about **ANY POINT** must be zero!

Engineering majors will take an entire course in static equilibrium:* **MME 211, Static Modeling of Mechanical Systems.

Doing Static Equilibrium Problems

Draw a free body diagram!

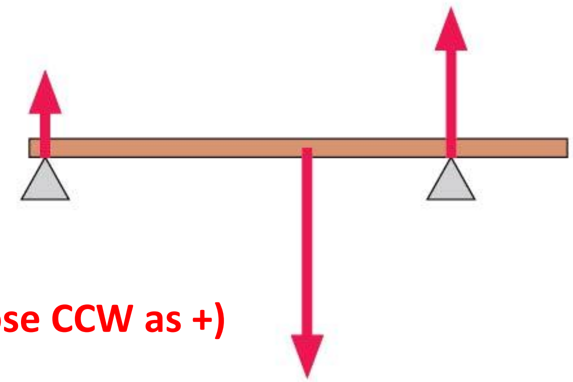
Now, you should draw the actual body. The weight of an object acts through its CM

MODEL 12.3

Static equilibrium

For extended objects at rest. **(not a point particle)**

- Model the object as a rigid body with no acceleration.
- Mathematically:
  - No net force: $\vec{F}_{net} = \sum \vec{F}_i = \vec{0}$, * and
  - No net torque: $\tau_{net} = \sum \tau_i = 0$ **(about any point, choose CCW as +)**
- The torque is zero about *every* point, so use any point that is convenient for the pivot point.
- Limitations: Model fails if either the forces or the torques aren't balanced.



$\vec{a} = \vec{0} \quad \alpha = 0$

The object is at rest.

* **Use in component form:**

$$\sum F_x = 0 \quad \sum F_y = 0$$

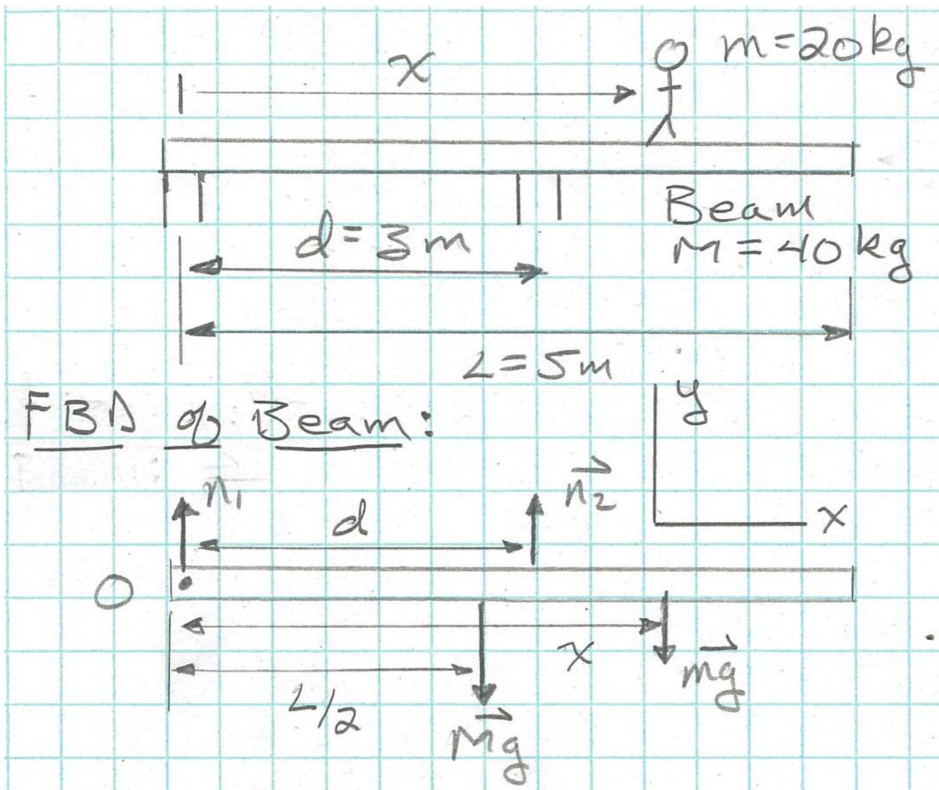
$$\sum F_z = 0 \text{ (if necessary)}$$

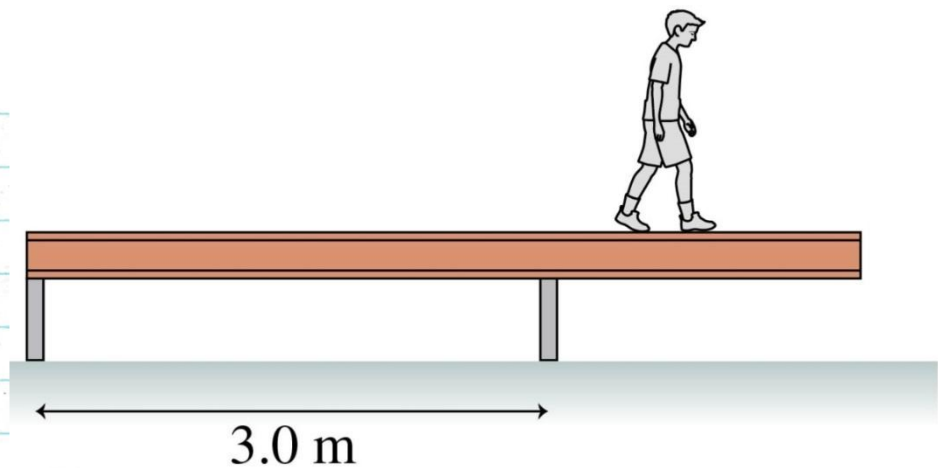
Whiteboard Problem 12-9

A 40 kg, 5.0m long beam is supported by, but not attached to, the two posts as shown. A 20 kg boy starts walking along the beam.

a) Sketch the problem and draw a free body diagram of the beam.

b) How close can he get to the right end of the beam without it falling over? (LC)





My Sketch and FBD:

A Comparison of Translation and Rotation Equations

Translation | Rotation

Kinematics (1D)

Translation: $$v = \frac{dx}{dt} \quad \& \quad a = \frac{dv}{dt}$$

Rotation: $$\omega = \frac{d\theta}{dt} \quad \& \quad \alpha = \frac{d\omega}{dt}$$

Integrated for Constant Acceleration

Translation	Rotation
$x_f = x_i + v_i \Delta t + \frac{1}{2} a (\Delta t)^2$	$\theta_f = \theta_i + \omega_i \Delta t + \frac{1}{2} \alpha (\Delta t)^2$
$v_f = v_i + a \Delta t$	$\omega_f = \omega_i + \alpha \Delta t$
$v_f^2 = v_i^2 + 2a(\Delta x)^2$	$\omega_f^2 = \omega_i^2 + 2\alpha(\Delta\theta)^2$

Dynamics: Newton's 2nd Law

Translation: $$\vec{F}_{net} = m\vec{a}$$

Rotation: $$\vec{\tau}_{net} = I\vec{\alpha}$$

We haven't done a full dynamics problem yet – let's do one now.

Kinetic Energy

Translation: $$K = \frac{1}{2} m v^2$$

Rotation: ??

Momentum

Translation: $$\vec{p} = m\vec{v}$$

Rotation: ??

Translation: $$\vec{p}_{\text{final}} = \vec{p}_{\text{initial}}$$

Rotation: ??

Whiteboard Problem 12-10

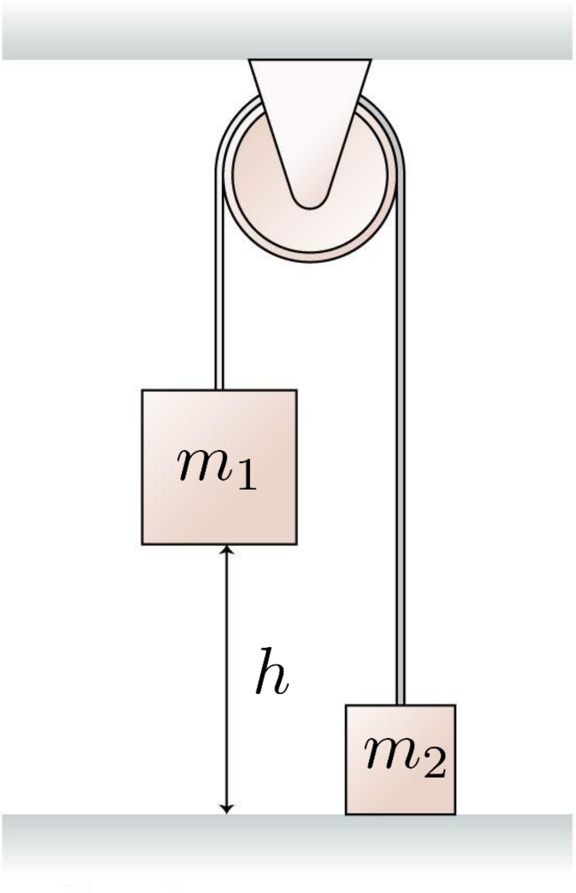
Mass m_1 = 4.0 kg and m_2 = 2.0 kg are connected over a pulley by a massless rope. The system is released from rest with m_1 a height h = 3 m above the floor. Find the time for m_1 to hit the floor.

a) **First, find the time of fall if the pulley is a massless ideal pulley. (LC)**

Note this part is straight out of Chap 7; don't need anything from Chap 12. Draw FBD's of the two blocks; impose constraints and solve. We'll do this one together as an example.

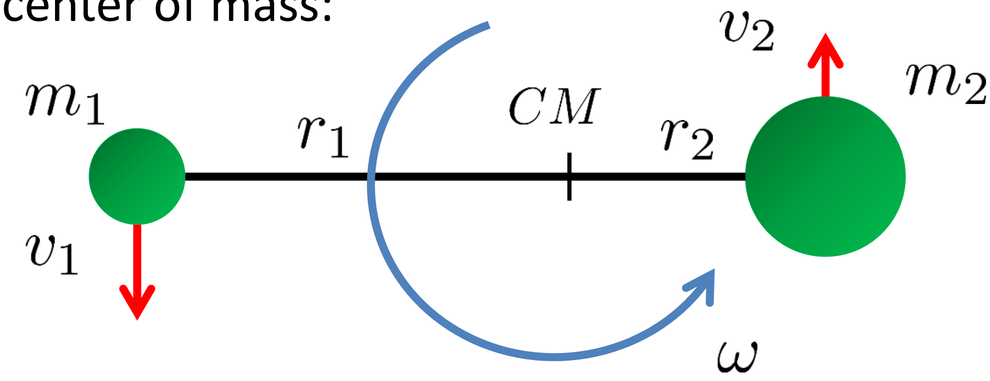
b) **Now treat the pulley as a real disk that is 12 cm in diameter with a mass of 8 kg, and find the time of fall. (LC, 2 pt. shot)**

Note, draw a FBD of the blocks and the pulley. For a real pulley, the tension in the rope will be different on the two sides of the pulley – otherwise, it wouldn't rotate. Apply Newton's 2nd for the blocks and the pulley; impose constraints, and solve.



Rotational Kinetic Energy

Consider two mass points connected by a massless rigid rod rotating about the center of mass:



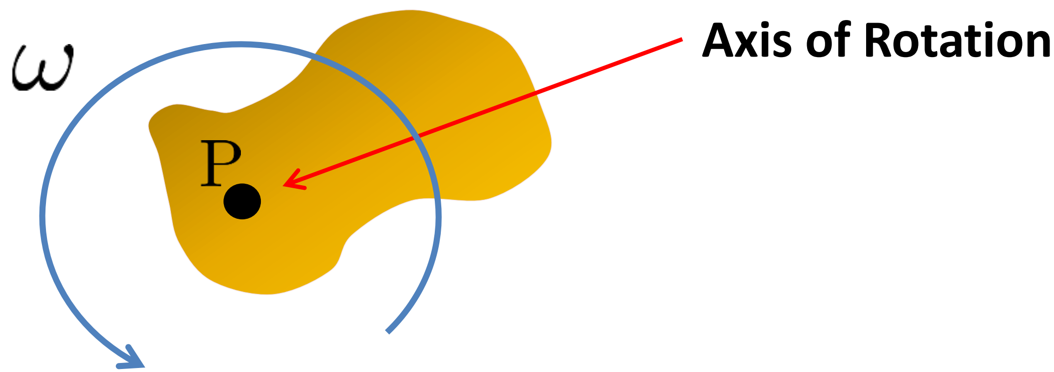
The total kinetic energy is:

$$K = \tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 v_2^2$$
$$= \tfrac{1}{2} m_1 r_1^2 \omega^2 + \tfrac{1}{2} m_2 r_2^2 \omega^2$$
$$= \tfrac{1}{2} \underbrace{(m_1 r_1^2 + m_2 r_2^2)}_{I,\ \text{the moment of inertia}} \omega^2$$

So, the Rotational Kinetic Energy is: $K_{\mathrm{rot}} = \tfrac{1}{2} I \omega^2$

Rotational Kinetic Energy

We can generalize this to any rotating rigid body:



Rotational Kinetic Energy, $K_{\text{rot}} = \frac{1}{2} I_p \omega^2$

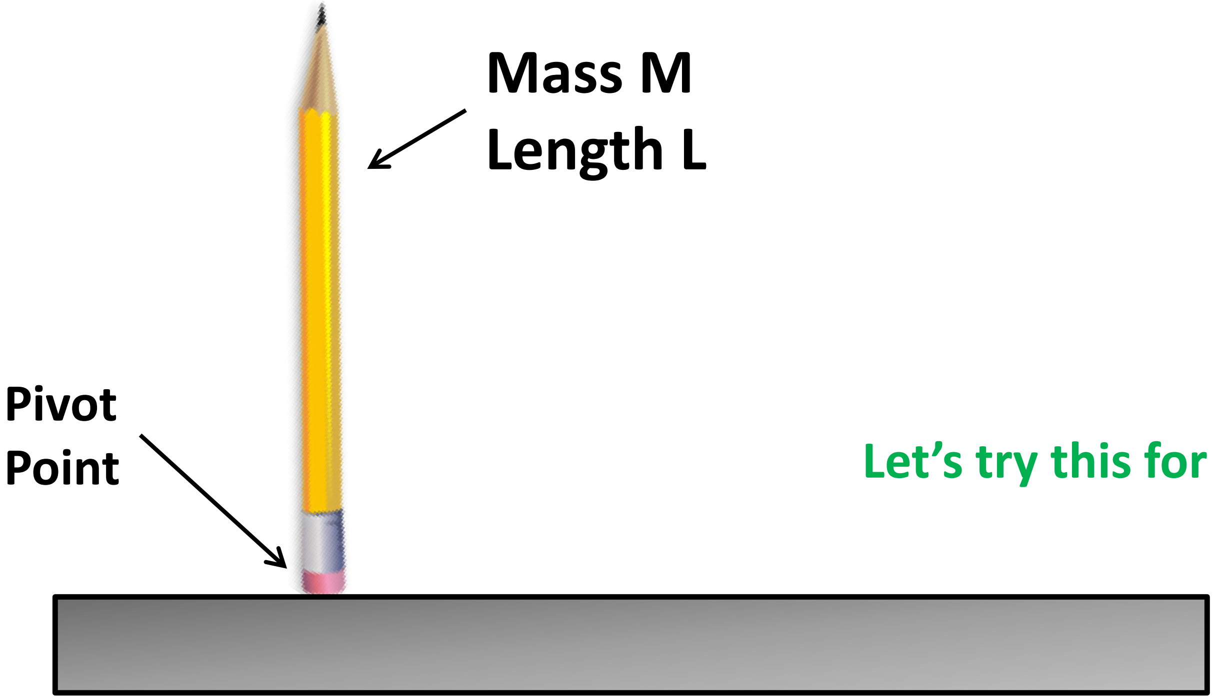
Where: I_p = Moment of inertia about the rotation axis, P

Compare this to $\frac{1}{2}mv^2$ for translational motion. Also note that a rigid body can have two forms of kinetic energy: translational, $\frac{1}{2}mv^2$, and rotational, $\frac{1}{2}I_p\omega^2$.

Whiteboard Problem: 12-11

A pencil has mass M and length L. It is standing straight up on a table on its eraser end. A slight push causes the pencil to fall over. Friction between the eraser and the table provides a pivot point for the pencil to rotate about.

<u>Use conservation of energy</u> to find an expression for the speed of the tip of the pencil as it hits the table. (LC, 2 pt. shot)



Let's try this for a typical pencil