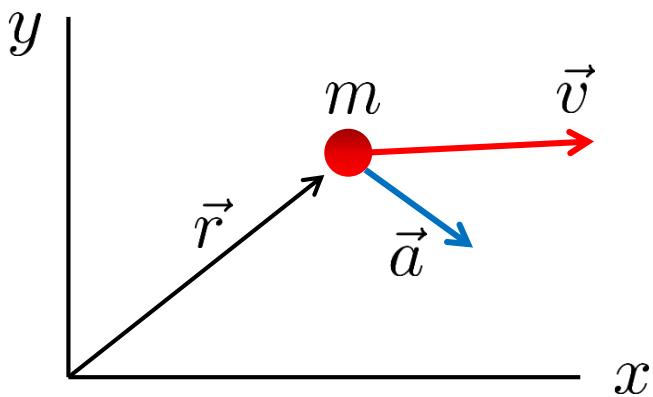
Chapter 12 - 1: Rotation of a Rigid Body

At the beginning of PHY181, we made the central observation that:

"Everything Moves!"

We have spent the entire time since then, analyzing this motion, but we have only considered one type of motion:

Translational Motion
(motion in 3D space)
Of a Point Particle
(picture an ice skater going back and forth across the ice)



But what else do ice skaters do? **They spin – or rotate.**

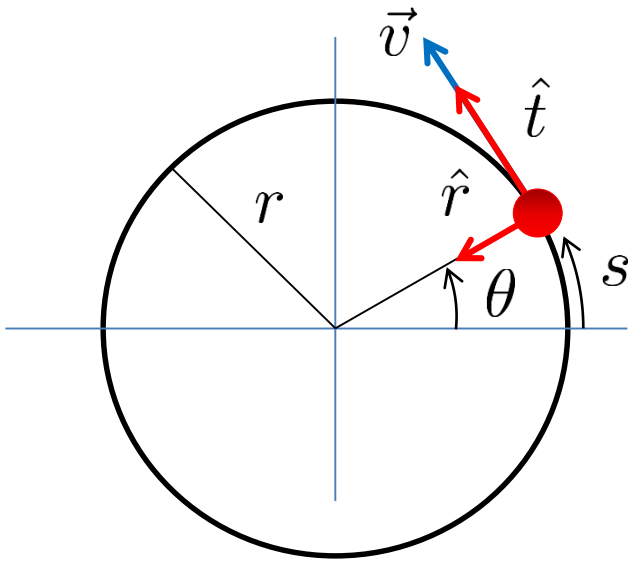
Real objects can have more motions than just translation. They can rotate (Chap 12), oscillate (i.e. vibrate) (Chap 15), and have wave motion (Chap 16). Many of the tools we have already developed can be used to treat these motions.

Warning: *Chap 12 is the longest in the book. Essentially, it repeats everything that we've done in 11 chapters for translational motion into one chapter for rotational motion. So, we'll do kinematics, dynamics, and conservation of energy and momentum – all for rotation, but for each of these, we have to define new ideas – consider Newton's 2nd.*

Some Stuff from Circular Motion

Uniform and Nonuniform Circular Motion:

In Chap 4 , we first looked at circular motion.
(*we skipped section 4.7 and 8.5 on this topic – go back and read them now*.)



Remember the radial ($\hat{r}$) and the tangential ($\hat{t}$) directions

For circular motion, we have the acceleration components:

$$a_r = \frac{v^2}{r}$$ (maintains the circular motion)

$$a_t = \frac{dv_t}{dt}$$ (changes the speed)

For $a_t =$ constant:

$$s_f = s_i + v_{t_i}\Delta t + \tfrac{1}{2}a_t(\Delta t)^2$$

$$v_{t_f} = v_{t_i} + a_t\Delta t$$

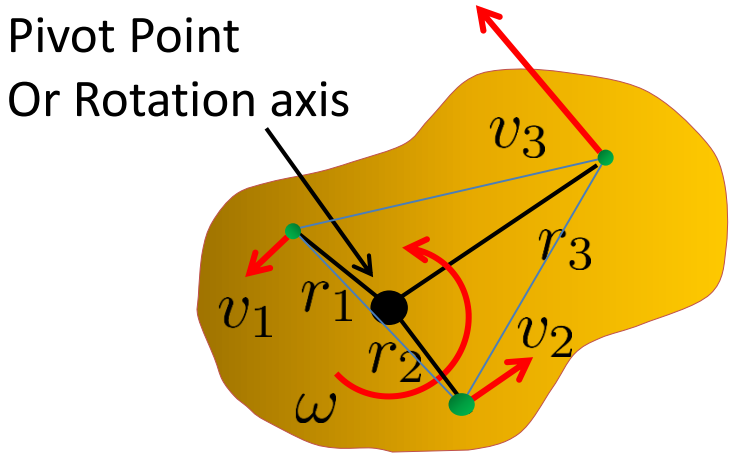
Or, using:

$$\theta = \frac{s}{r} \qquad \theta_f = \theta_i + \omega_i\Delta t + \tfrac{1}{2}\tfrac{a_t}{r}(\Delta t)^2$$

$$\omega = \frac{v_t}{r} \qquad \omega_f = \omega_i + \tfrac{a_t}{r}\Delta t$$

These types of equations can be used to describe the **rotation of a rigid body.**

Rigid Body Rotation



What is a rigid body?

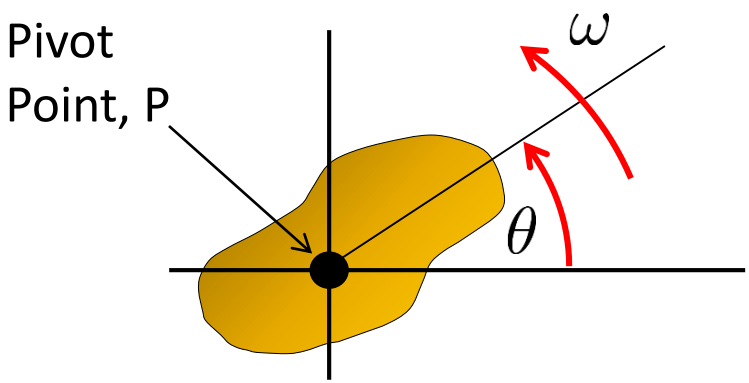
A rigid body is an extended object whose <u>components don't move relative to each other</u>.

Can be a finite collection of particles at fixed positions or an infinite collection – i.e. a continuous body.

If the body is pivoted at some point, it is free to rotate around that point or about an axis through that point.

All points in the body have the same $\omega = \frac{v}{r}$, but different v

Rotational Kinematics



Basic Definitions:

$$\text{Angular Position} = \theta$$
$$\text{Angular Velocity} = \omega = \frac{d\theta}{dt}$$
$$\text{Angular Acceleration} = \alpha = \frac{d\omega}{dt}$$

Sign Convention:

CounterClockWise (CCW) is positive.

Rotational Kinematics for Constant Acceleration:

$$\text{For } \alpha = \frac{a_t}{r} = \text{constant between times } t_i \text{ and } t_f\text{:}$$

$$\theta_f = \theta_i + \omega_i \Delta t + \tfrac{1}{2}\alpha(\Delta t)^2$$
$$\omega_f = \omega_i + \alpha \Delta t \qquad \Delta t = t_f - t_i$$
$$\omega_f^2 = \omega_i^2 + 2\alpha\Delta\theta \qquad \Delta\theta = \theta_f - \theta_i$$

These equations should look familiar. They're the same that we used for 1D constant acceleration kinematics way back in Chap 2, but here with angular quantities!

Whiteboard Problem 12-1

An ice skater holds her arms outstretched as she spins at 180 rpm.
What is the speed of her hands if they are 140 cm apart? (LC)

In rotational problems, you do have to be careful with your units. Angular position, velocity, and acceleration can be used in a variety of units; <u>however, when you use an angular quantity to calculate a linear quantity, the angular quantity must be in radians</u>.

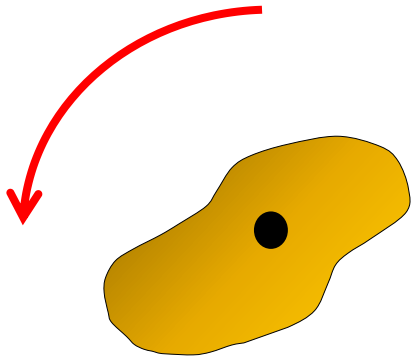
$$\text{e.g. in } v = r\omega,\ \omega \text{ must be in radians/sec.}$$

Whiteboard Problem 12-2

A high speed drill starts from rest and reaches 2000 rpm in a time of 0.50 s.

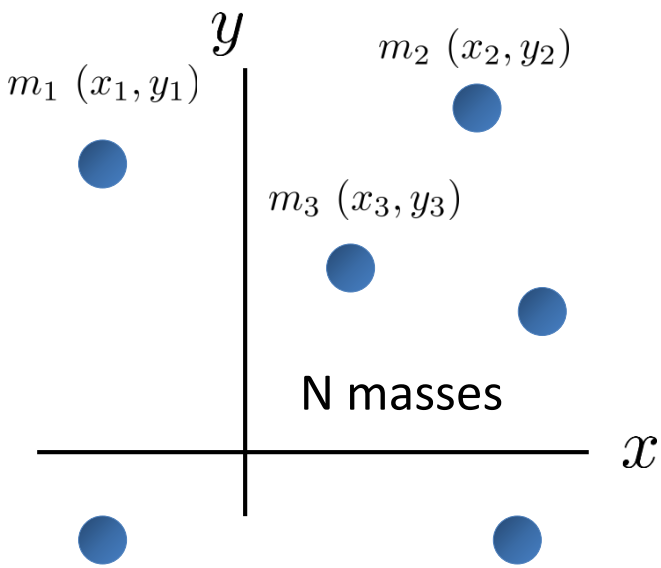
a) **What is the drill's acceleration (assumed to be constant)?(LC)**
a) **Through how many revolutions does it turn this first 0.5 s? (LC)**

Center of Mass



The center of mass is the point about which an unconstrained rigid body will rotate.

Center of Mass for a Discrete Collection of Point Masses:



Center of Mass Coordinates:

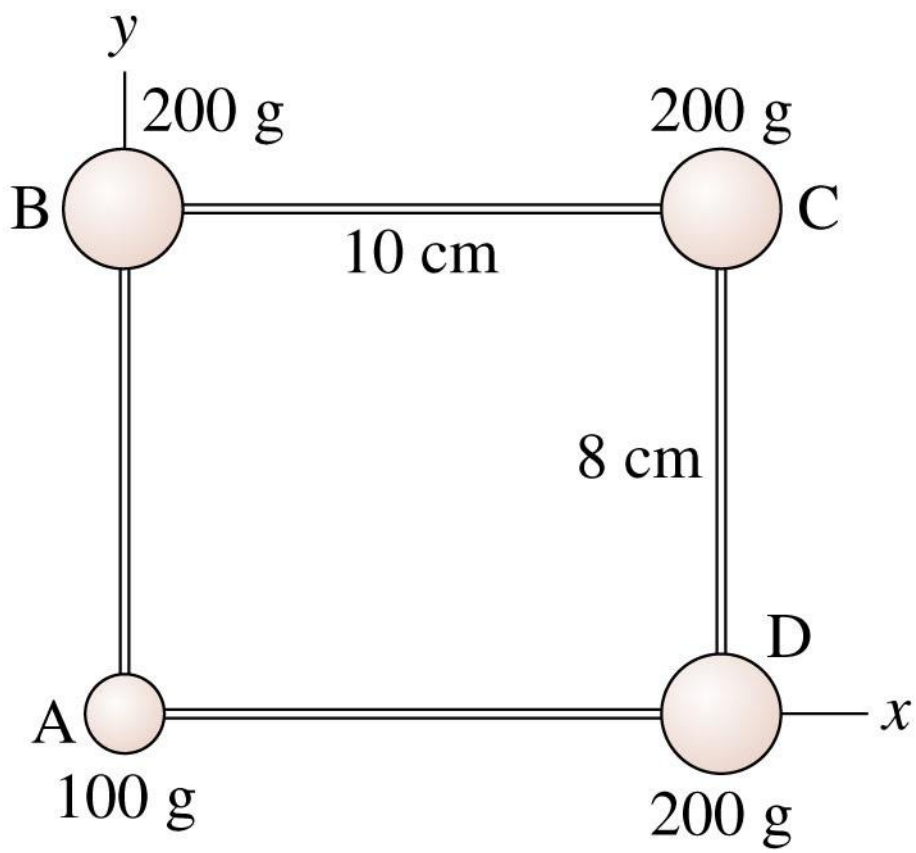
$$x_{cm} = \frac{1}{M}\sum_{i=1}^{N} m_i x_i$$

$$y_{cm} = \frac{1}{M}\sum_{i=1}^{N} m_i y_i$$

$$\text{where}: M = \text{Total Mass} = \sum_{i=1}^{N} m_i$$

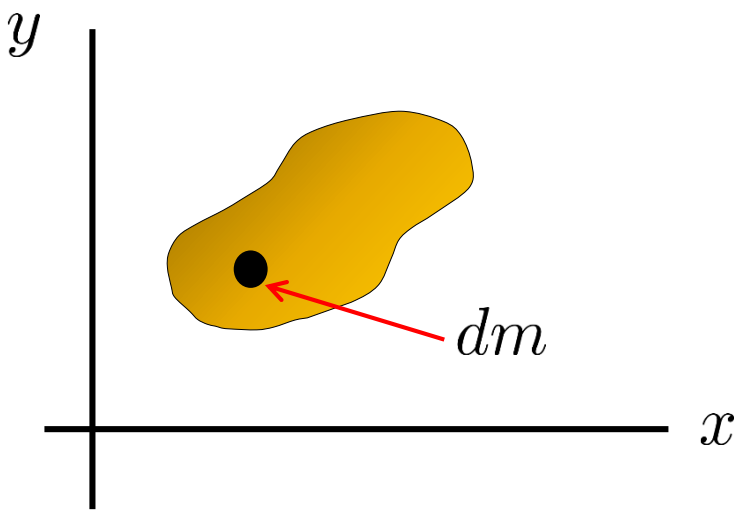
Whiteboard Problem 12-3

The masses shown below are connected by massless rigid rods.
Find the x (LC) and y coordinates of the center of mass.



Does it matter what units you work in for this problem?

Quick Note about Continuous Bodies



The center of mass coordinates are at:

$$x_{cm} = \frac{1}{M}\int x dm$$

$$y_{cm} = \frac{1}{M}\int y dm$$

These can become nasty integrals - **we won't do any!** But if you take Engineering Mechanics, you'll do a lot.

Your author makes the important point that **for continuous bodies that are symmetric and have uniform density, the center of mass is at the geometric center.**

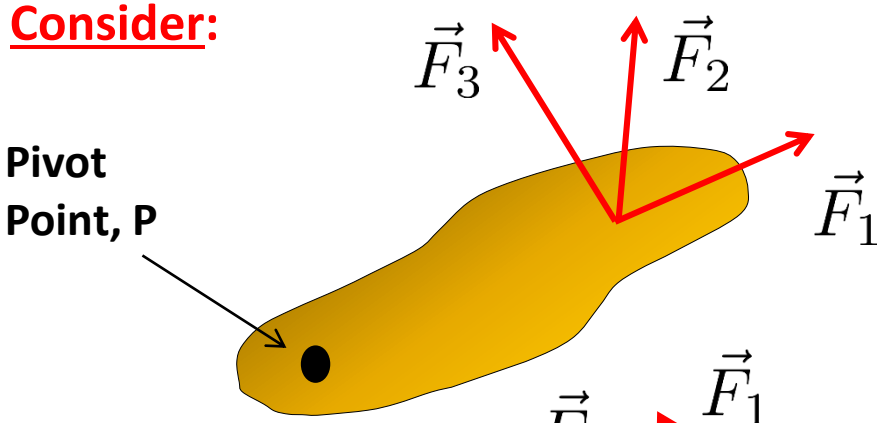
Torque

Torque (also called "moment" in engineering) is the **rotational analog of force**.

An applied force on an object causes an acceleration: $\vec{a} = \dfrac{\vec{F}}{m}$

An applied force on a rigid body can create a **torque that causes an angular acceleration about some point**. The torque depends on the size of the force, but what else?

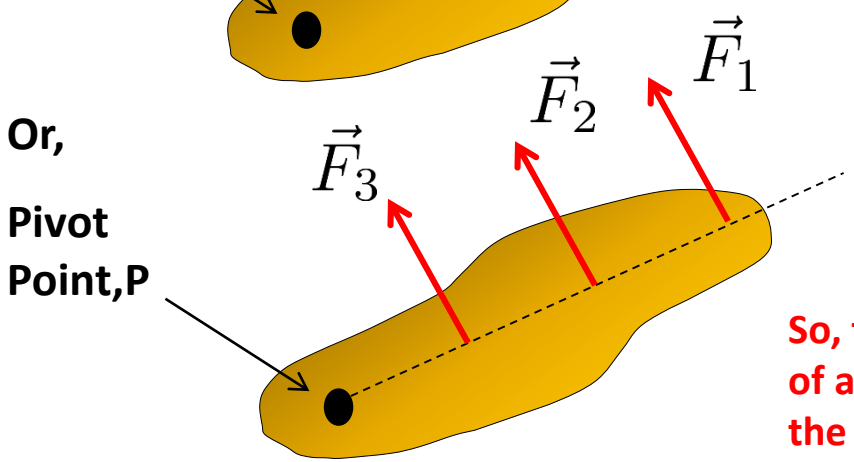
Consider:



The forces have equal magnitudes. Which force causes the most rotation about the pivot point? (LC)

$$(\text{answer: } \vec{F}_3)$$

Or,

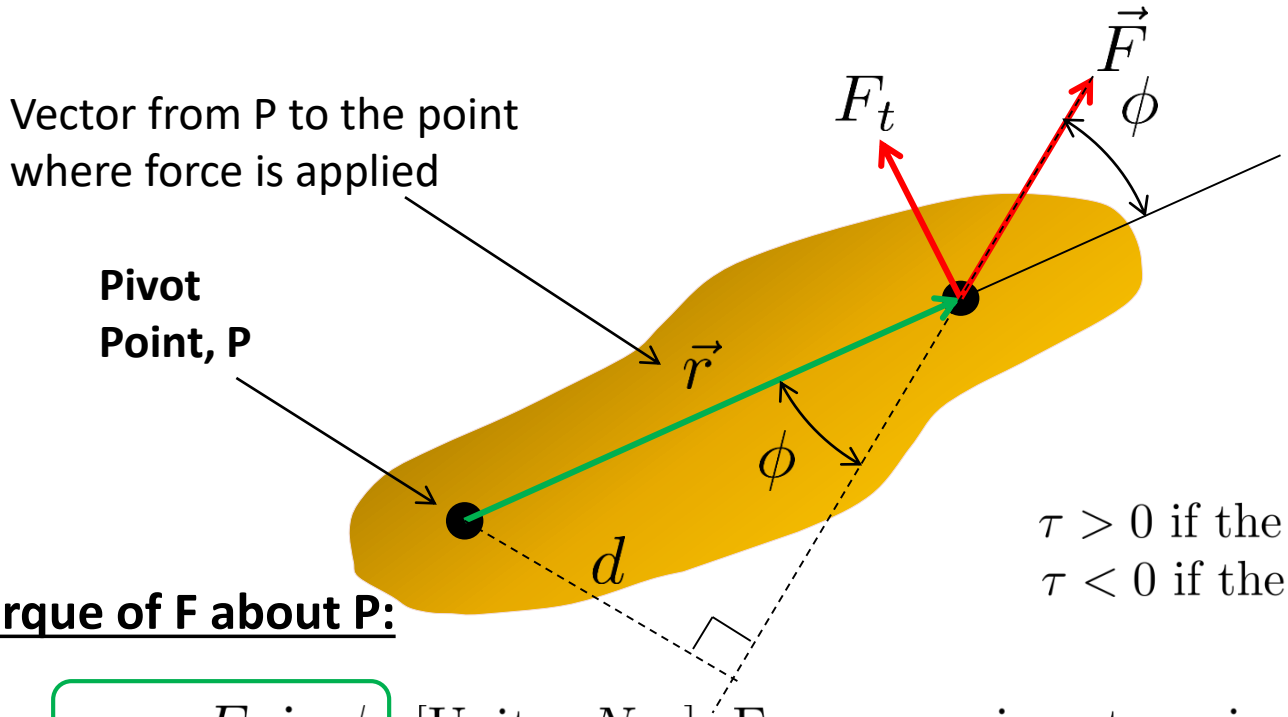


The forces have equal magnitudes. Which force causes the most rotation about the pivot point? (LC)

$$(\text{answer: } \vec{F}_1)$$

So, the greater the distance from P to the point of application of the force and the more perpendicular the force is to the line between P and the force, the more rotation.

Two Ways to Calculate the Torque Produced by a Force



$\tau > 0$ if the rotation is CCW
$\tau < 0$ if the rotation is CW

Torque of F about P:

$\boxed{\tau = rF\sin\phi}$ [Units: Nm] From experiments as in the previous slide

1.) $\tau = rF_t$; where: F_t = tangential component of $\vec{F}$ ($\perp \vec{r}$) $= F\sin\phi$

So, $\boxed{\tau = rF\sin\phi}$

2.) $\tau = dF$ where d = the perpendicular distance from P to the line of $\vec{F}$

(*torque arm or moment arm*)

$$d = r\sin\phi$$

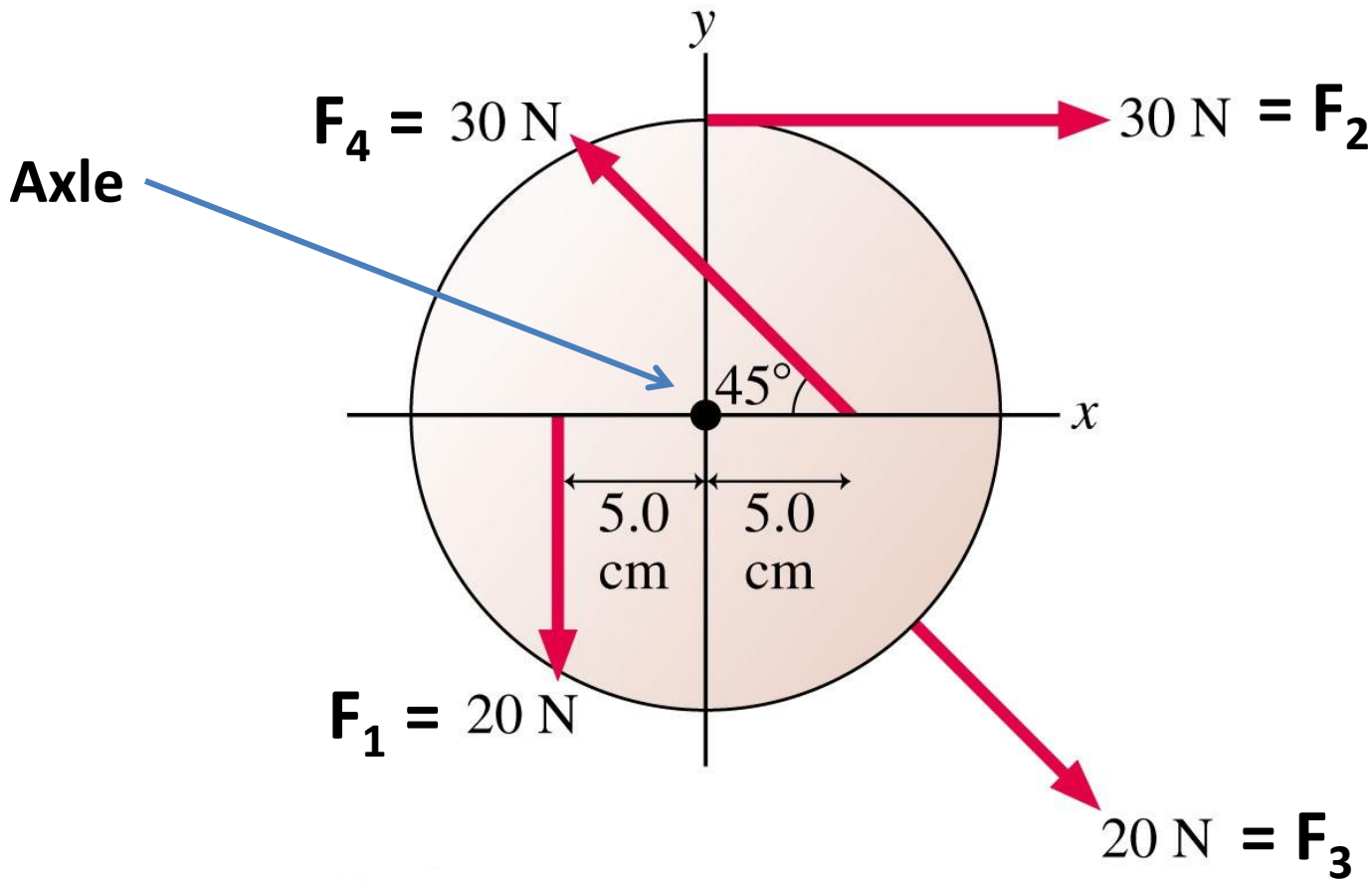
So, $\boxed{\tau = rF\sin\phi}$

There is a more general way to calculate the torque produced by a force; we'll get to it in a few minutes.

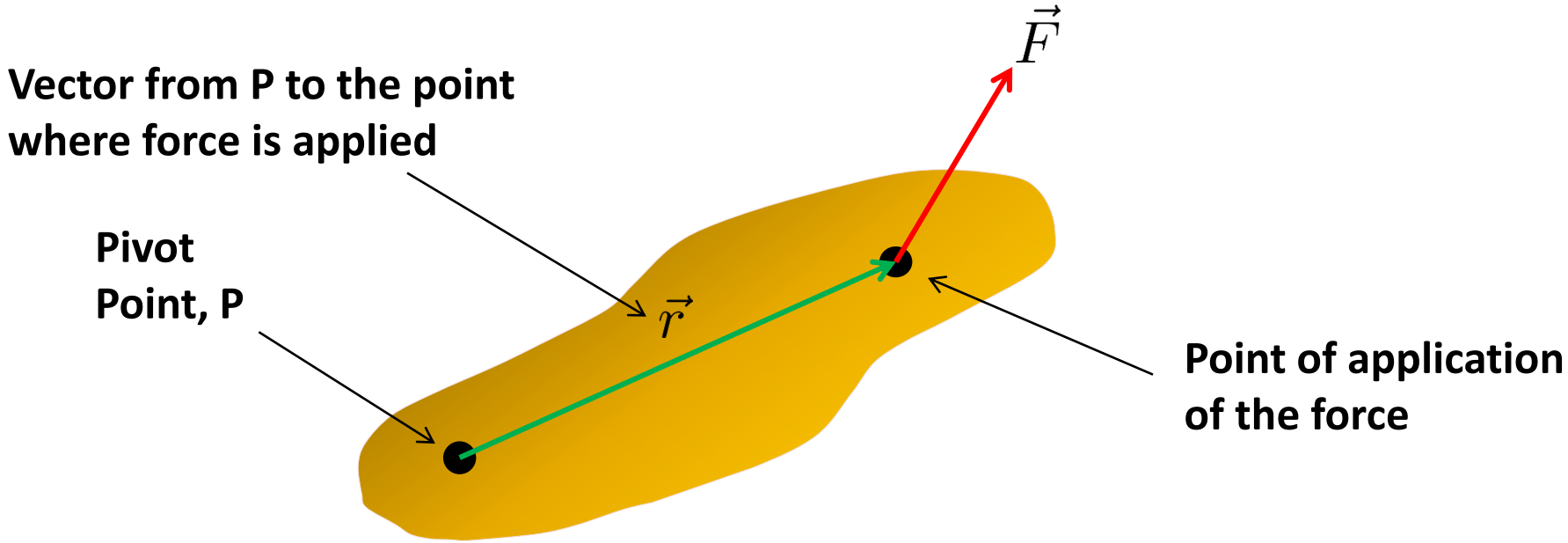
Whiteboard Problem 12-4

The 20 cm diameter disk shown below can rotate about an axle through its center.

What is the net torque about the axle? (LC)



General Definition of Torque Produced by a Force



Remember when we were first calculating the work done by a force, and the vector dot product gave us an easy way to calculate the work that automatically included only the component of the force parallel to the displacement (dot product)?

There's something similar for Torque:

The Torque produced by the force F <u>about the point P</u> is:

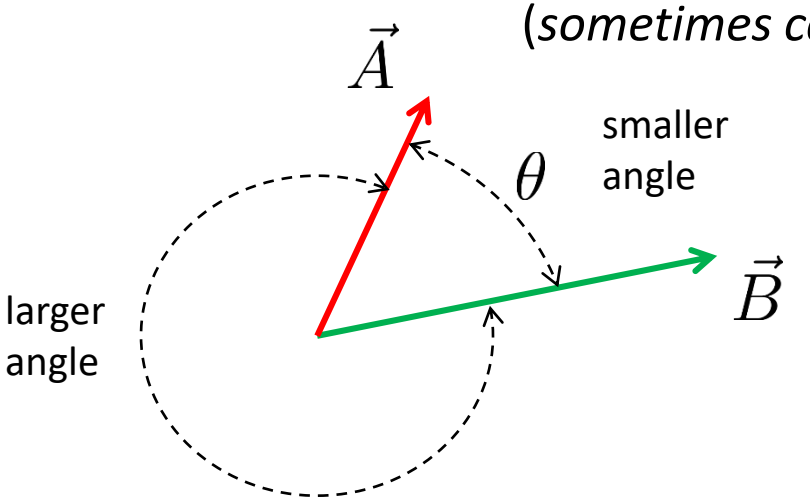
$$\text{Torque of } \vec{F} \text{ about P:} \quad \vec{\tau} = \vec{r} \times \vec{F}$$

What is this?

It's called a Vector Cross Product.

Vector Cross Product

(*sometimes called a Vector Product*)



Vector, $\vec{C} = \vec{A} \times \vec{B}$

Where: $|\vec{C}| = AB \sin\theta$

Direction of $\vec{C}$ is from the Right Hand Rule (RHR)

Where: A and B are the magnitudes of vectors $\vec{A}$ and $\vec{B}$

θ is the smaller of the two angles between $\vec{A}$ and $\vec{B}$

$$\boxed{\text{So, } \vec{C} = \vec{A} \times \vec{B} = (AB \sin\theta, \text{direction by RHR})}$$

What does the cross product mean in words?

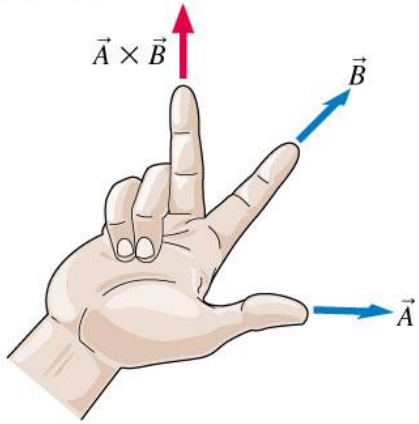
$\vec{A} \times \vec{B}$ is a measure of how much of $\vec{A}$ is perpendicular to $\vec{B}$ and in what direction, and vice versa.

What is the RHR - The Right Hand Rule?

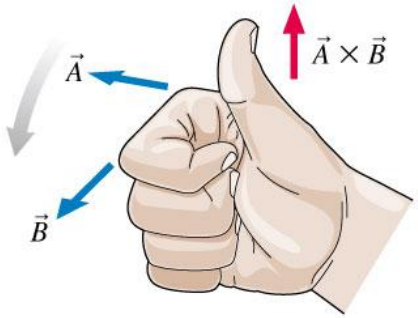
The Right Hand Rule (From your text)

Using the right-hand rule

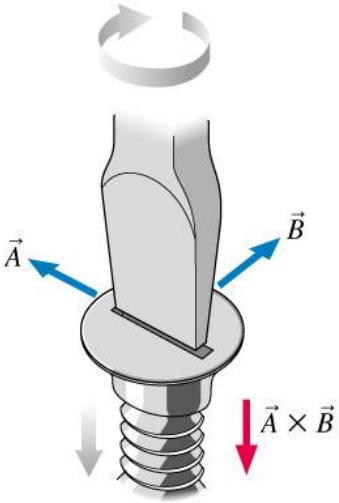
Spread your *right* thumb and index finger apart by angle α. Bend your middle finger so that it is *perpendicular* to your thumb and index finger. Orient your hand so that your thumb points in the direction of $\vec{A}$ and your index finger in the direction of $\vec{B}$. Your middle finger now points in the direction of $\vec{A} \times \vec{B}$.



Make a loose fist with your *right* hand with your thumb extended outward. Orient your hand so that your thumb is perpendicular to the plane of $\vec{A}$ and $\vec{B}$ and your fingers are curling *from* the line of vector $\vec{A}$ *toward* the line of vector $\vec{B}$. Your thumb now points in the direction of $\vec{A} \times \vec{B}$.



Imagine using a screwdriver to turn the slot in the head of a screw from the direction of $\vec{A}$ to the direction of $\vec{B}$. The screw will move either "in" or "out." The direction in which the screw moves is the direction of $\vec{A} \times \vec{B}$.

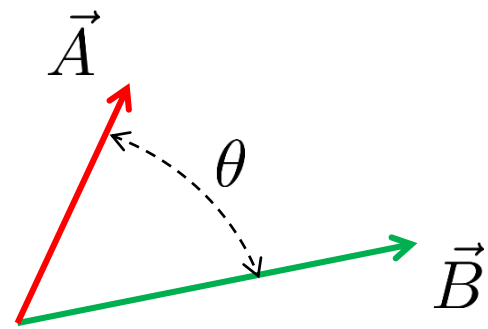


There are many ways to do the RHR. The way that I do the Right Hand Rule is a variation of this way.

The Right Hand Rule (My Way)

1. **Use your right hand!**

$$\vec{C} = \vec{A} \times \vec{B}$$



2. With your four fingers extended and together, point them in the direction of the first vector of the cross product.

3. Rotate your hand about an axis through your forearm until you can close your fingers through the angle θ towards the second vector in the product.

4. Your thumb points in the direction of the cross product.

So, $\vec{C} = \vec{A} \times \vec{B}$ points into the screen.

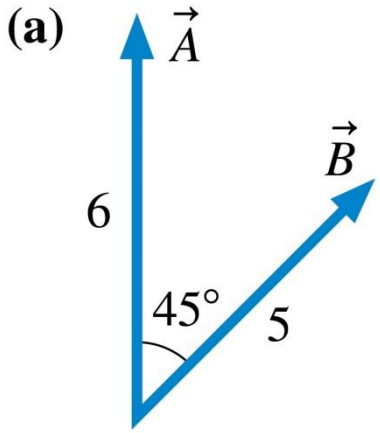
And, $\vec{D} = \vec{B} \times \vec{A}$ points out of the screen.

Note: $\vec{A} \times \vec{B}$ is perpendicular to both $\vec{A}$ and $\vec{B}$.

And, $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$

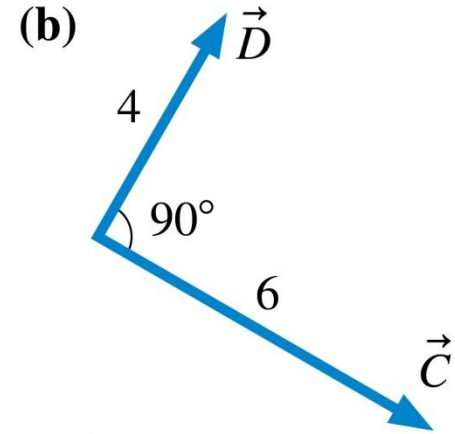
WB Problem: 12-5 Some Cross Products

Evaluate the following cross products (give direction as in or out):



$\vec{A} \times \vec{B}$

LC (mag only)
Ans: (21.21, in)



$\vec{C} \times \vec{D}$

LC (mag only)
Ans: (24.0,out)

(c) Find cross product in <u>Component Form</u>.

$$\vec{A} = 3\hat{\imath} + \hat{\jmath} \qquad \vec{B} = 3\hat{\imath} - 2\hat{\jmath} + 2\hat{k}$$

Find: $\vec{A} \times \vec{B}$

Hmmm, maybe we need another way to do a Cross Product

Vector Cross Product using a Determinant

When you know only the components of the vectors in a cross product, it may difficult to find the angle between them, and it can be even more difficult to apply the RHR and find the components of the cross product vector. **Here's how you do it easily:**

A determinant is something that you'll learn about in a Math course, perhaps linear algebra. All we need to know now is how to expand one. **A 2 X 2 determinant is the simplest (a,b,c,d are just numbers):**

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$$

Note, the negative sign; when you expand a determinant, the signs on the terms always alternate.

Now a 3 X 3 determinant can be expanded into three 2 X 2 determinants:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \boxed{a\begin{vmatrix} e & f \\ h & i \end{vmatrix}} - \boxed{b\begin{vmatrix} d & f \\ g & i \end{vmatrix}} + \boxed{c\begin{vmatrix} d & e \\ g & h \end{vmatrix}}$$

(Now, just expand the 2 X 2's) $= a(ei - hf) - b(di - gf) + c(dh - ge)$

Note, there are many ways to expand a determinant. If you know one that you like and it works, use that technique.

OK, how does this work for a vector cross product?

Vector Cross Product using a Determinant

For the vectors:

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$$

$$\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$$

$$\boxed{\vec{C} = \vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}} = \hat{\imath}\begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} - \hat{\jmath}\begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} + \hat{k}\begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}$$

$$= \hat{\imath}\underbrace{(A_yB_z - B_yA_z)}_{C_x} - \hat{\jmath}\underbrace{(A_xB_z - B_xA_z)}_{C_y = (B_xA_z - A_xB_z)} + \hat{k}\underbrace{(A_xB_y - B_xA_y)}_{C_z}$$

So, this method directly gives the components of the cross product vector. You don't need to know any angles, and there is no Right Hand Rule! See the Equation Sheet.

Now, Let's go back to Problem WB 12-5c and do this one together

(c) Find cross product in Component Form. (LC)

$$\vec{A} = 3\hat{\imath} + \hat{\jmath} \qquad \vec{B} = 3\hat{\imath} - 2\hat{\jmath} + 2\hat{k}$$

Find: $\vec{A} \times \vec{B}$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & 1 & 0 \\ 3 & -2 & 2 \end{vmatrix}$$

$$= \hat{\imath} \begin{vmatrix} 1 & 0 \\ -2 & 2 \end{vmatrix} - \hat{\jmath} \begin{vmatrix} 3 & 0 \\ 3 & 2 \end{vmatrix} + \hat{k} \begin{vmatrix} 3 & 1 \\ 3 & -2 \end{vmatrix}$$

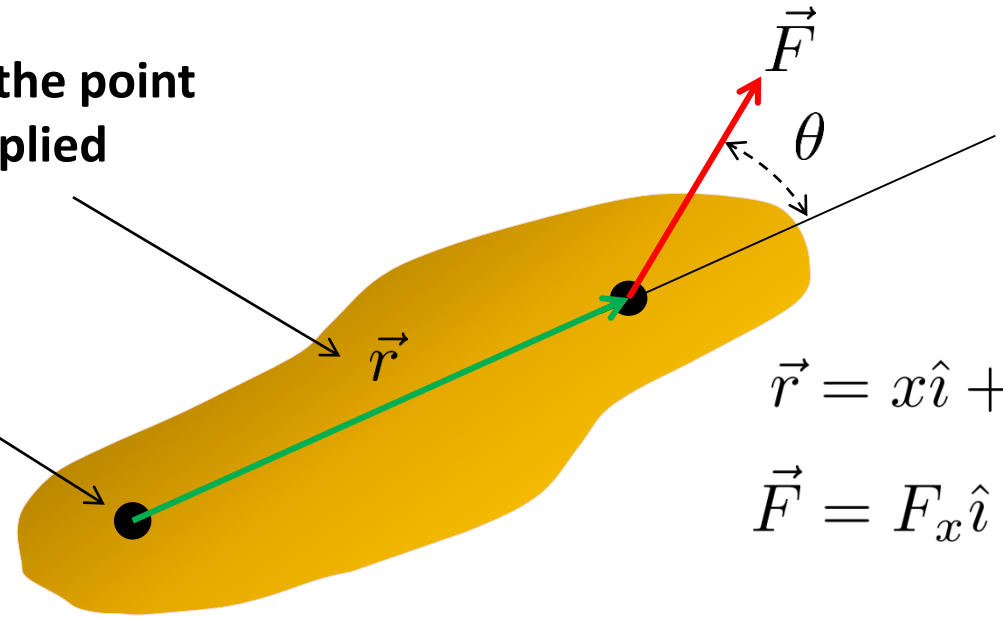
$$= \hat{\imath}[(1)(2) - (-2)(0)] - \hat{\jmath}[(3)(2) - (3)(0)] + \hat{k}[(3)(-2) - (3)(1)]$$

So, $\vec{A} \times \vec{B} = 2\hat{\imath} - 6\hat{\jmath} - 9\hat{k}$ and, $|\vec{A} \times \vec{B}| = 11.0$

Back to Torque

Vector from P to the point where force is applied

Pivot Point, P



$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

$$\vec{F} = F_x\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k}$$

Now, we have two ways to calculate the **Torque of F about the point P**:

$$\text{Torque:} \quad \vec{\tau} = \vec{r} \times \vec{F} = (rF\sin\theta, \text{direction by RHR})$$

$$\text{Torque:} \quad \vec{\tau} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$$

Which one do we use?
It depends on the problem.
You have to able to use both!

Whiteboard Problem 12-6

A force $\vec{F} = -10\hat{\jmath}\ N$ is exerted on a particle at $\vec{r} = 5\hat{\imath} + 5\vec{\jmath}\ m$.

a) Draw x, y, z axes and show $\vec{F}$ and $\vec{r}$ (Make sure you use a right–handed (x, y, z) coordinate system where $\hat{\imath} \times \hat{\jmath} = \hat{k}$.)

b) Find the torque on the particle about the origin using $\vec{A} \times \vec{B} = \{AB \sin\theta, RHR\}$. **(Enter torque <u>magnitude</u> in LC)**

c) Find the torque on the particle about the origin using just the components. **(Enter torque <u>vector</u> in LC)**